

## CYCLIC SIEVING AND PLETHYSM COEFFICIENTS

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ABSTRACT. A combinatorial expression for the coefficient of the Schur function  $s_\lambda$  in the expansion of the plethysm  $p_{n/d}^d \circ s_\mu$  is given for all  $d$  dividing  $n$  for the cases in which  $n = 2$  or  $\lambda$  is rectangular. In these cases, the coefficient  $\langle p_{n/d}^d \circ s_\mu, s_\lambda \rangle$  is shown to count, up to sign, the number of fixed points of an  $\langle s_\mu^n, s_\lambda \rangle$ -element set under the  $d^{\text{th}}$  power of an order  $n$  cyclic action. If  $n = 2$ , the action is the Schützenberger involution on semistandard Young tableaux (also known as evacuation), and, if  $\lambda$  is rectangular, the action is a certain power of Schützenberger and Shimozono’s *jeu-de-taquin* promotion.

This work extends results of Stembridge and Rhoades linking fixed points of the Schützenberger actions to ribbon tableaux enumeration. The conclusion for the case  $n = 2$  is equivalent to the domino tableaux rule of Carré and Leclerc for discriminating between the symmetric and antisymmetric parts of the square of a Schur function.

## 1. INTRODUCTION

A principal concern of algebraic combinatorics is the identification of collections of combinatorial objects that occur in algebraically significant multiplicities. Perhaps the most celebrated success in this endeavor is the Littlewood–Richardson rule, which gives a combinatorial description for the coefficient of each Schur function arising in the expansion of a product of Schur functions on the Schur basis. For the case of a Schur function  $s_\mu$  raised to the  $n^{\text{th}}$  power, there is a natural order  $n$  cyclic action on the objects specified by the Littlewood–Richardson rule for the coefficient of  $s_\lambda$ , provided that  $n = 2$  or  $\lambda$  is rectangular. In this article, we present an algebraic expression for the number of fixed points under each power of this cyclic action, à la the cyclic sieving phenomenon of Reiner, Stanton, and White [19]. In particular, we show that the cardinality of each fixed point set is given up to sign by the coefficient of  $s_\lambda$  in the expansion of a plethysm involving  $s_\mu$ . Since the plethysm corresponding to the trivial action is  $s_\mu^n$ , what we put forth may be viewed as an *accoutrement* to the Littlewood–Richardson rule that endows a series of associated collections of objects with algebraic meaning, and, in so doing, underscores the power of the cyclic sieving paradigm.

Let  $\Lambda$  be the ring of symmetric functions over  $\mathbb{Z}$ . For all  $f, g \in \Lambda$ , if  $V$  and  $W$  are polynomial representations of  $GL_m(\mathbb{C})$  with characters  $\chi_V =$

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$f(x_1, x_2, \dots, x_m)$  and  $\chi_W = g(x_1, x_2, \dots, x_m)$ , respectively, then  $\chi_{V \oplus W} = (f + g)(x_1, x_2, \dots, x_m)$  and  $\chi_{V \otimes W} = (fg)(x_1, x_2, \dots, x_m)$ . Plethysm is a binary operation on  $\Lambda$  (so named by Littlewood [17] in 1950) that is compatible with representation composition in the same sense that addition and multiplication correspond to representation direct sum and tensor product, respectively. To wit, if  $\rho: GL_m(\mathbb{C}) \rightarrow GL_M(\mathbb{C})$  is a polynomial representation of  $GL_m(\mathbb{C})$  with character  $g(x_1, x_2, \dots, x_m)$ , and  $\sigma: GL_M(\mathbb{C}) \rightarrow GL_N(\mathbb{C})$  is a polynomial representation of  $GL_M(\mathbb{C})$  with character  $f(x_1, x_2, \dots, x_M)$ , then the composition  $\sigma\rho: GL_m(\mathbb{C}) \rightarrow GL_N(\mathbb{C})$  is a polynomial representation of  $GL_m(\mathbb{C})$  with character  $(f \circ g)(x_1, x_2, \dots, x_m)$ , where  $f \circ g \in \Lambda$  denotes the plethysm of  $f$  and  $g$ . A formal definition is given in section 2.

We are herein concerned with plethysms of the form  $p_{n/d}^d \circ s_\mu$ , where  $\mu$  is a partition,  $s_\mu$  denotes the Schur function associated to  $\mu$ ,  $d$  divides  $n$ , and  $p_{n/d}$  denotes the  $(n/d)^{\text{th}}$  power-sum symmetric function,  $x_1^{n/d} + x_2^{n/d} + \dots$ . Defining an inner product  $\langle \cdot, \cdot \rangle$  on  $\Lambda$  by requiring that the Schur functions form an orthonormal basis, we obtain a convenient notation —  $\langle f, s_\lambda \rangle$  — for the coefficient of  $s_\lambda$  in the expansion of a symmetric function  $f$  as a linear combination of Schur functions. The main achievement in this article is a combinatorial description of the coefficients  $\langle p_{n/d}^d \circ s_\mu, s_\lambda \rangle$  for the cases in which  $n = 2$  or  $\lambda$  is rectangular.

Let  $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ . If  $n = 2$ , the Littlewood–Richardson multiplicity  $\langle s_\mu^n, s_\lambda \rangle$  is the number of semistandard Young tableaux of shape  $\lambda$  and content  $\bar{\mu}\mu := (\mu_m, \dots, \mu_1, \mu_1, \dots, \mu_m)$  for which the reading word is anti-Yamanouchi in  $\{1, 2, \dots, m\}$  and Yamanouchi in  $\{m + 1, m + 2, \dots, 2m\}$ . The Schützenberger involution (also known as evacuation) on a semistandard tableau preserves the shape and reverses the content, so it gives an action on the tableaux of shape  $\lambda$  and content  $\bar{\mu}\mu$ , which turns out to restrict to those tableaux with words satisfying the aforementioned Yamanouchi conditions (cf. Remark 4.21).

For the case in which  $n$  may vary, we treat the coefficient  $\langle s_\mu^n, s_\lambda \rangle$  somewhat differently. In general, the Littlewood–Richardson multiplicity  $\langle s_\mu^n, s_\lambda \rangle$  is the number of semistandard Young tableaux of shape  $\lambda$  and content  $\mu^n := (\mu_1, \dots, \mu_m, \mu_1, \dots, \mu_m, \dots, \mu_1, \dots, \mu_m)$  for which the reading word is Yamanouchi in the alphabets  $\{km + 1, km + 2, \dots, (k + 1)m\}$  for all  $0 \leq k \leq n - 1$ . On a semistandard tableau, *jeu-de-taquin* promotion (also introduced by Schützenberger; cf. [22]) preserves the shape and permutes the content by the long cycle in  $\mathfrak{S}_{mn}$ , so  $m$  iterations of promotion gives an action on the tableaux of shape  $\lambda$  and content  $\mu^n$ . If  $\lambda$  is rectangular, this action has order  $n$ , and it, too, restricts to those tableaux with words satisfying the requisite Yamanouchi conditions (cf. Remark 4.31).

We are at last poised to state our main results.

**Theorem 1.1.** *Let  $\text{EYTab}(\lambda, \bar{\mu}\mu)$  be the set of all semistandard tableaux of shape  $\lambda$  and content  $\bar{\mu}\mu$  with reading word anti-Yamanouchi in  $\{1, 2, \dots, m\}$  and Yamanouchi in  $\{m + 1, m + 2, \dots, 2m\}$ , and let  $\xi$  act on  $\text{EYTab}(\lambda, \bar{\mu}\mu)$*

by the Schützenberger involution. Then

$$|\{T \in \text{EYTab}(\lambda, \bar{\mu}\mu) : \xi(T) = T\}| = \pm \langle p_2 \circ s_\mu, s_\lambda \rangle.$$

**Theorem 1.2.** *Let  $\lambda$  be a rectangular partition, and let  $\text{PYTab}(\lambda, \mu^n)$  be the set of all semistandard tableaux of shape  $\lambda$  and content  $\mu^n$  with reading word Yamanouchi in the alphabets  $\{km + 1, km + 2, \dots, (k + 1)m\}$  for all  $0 \leq k \leq n - 1$ . Let  $j$  act on  $\text{PYTab}(\lambda, \mu^n)$  by  $m$  iterations of jeu-de-taquin promotion. Then, for all positive integers  $d$  dividing  $n$ ,*

$$|\{T \in \text{PYTab}(\lambda, \mu^n) : j^d(T) = T\}| = \pm \langle p_{n/d}^d \circ s_\mu, s_\lambda \rangle.$$

From Theorems 3.1 and 3.2 in Lascoux–Leclerc–Thibon [14], we see that the Hall–Littlewood symmetric function  $Q'_{1n}(q)$  specializes (up to sign) at  $q = e^{\frac{2\pi i \ell}{n}}$  to  $p_{n/\gcd(n, \ell)}^{\gcd(n, \ell)}$ . Therefore, we may interpret Theorem 1.2 as analogous to exhibiting an instance of the cyclic sieving phenomenon.

**Corollary 1.3.** *Let  $\lambda$  be a rectangular partition. Let  $j$  act on  $\text{PYTab}(\lambda, \mu^n)$  by  $m$  iterations of jeu-de-taquin promotion. Then, for all integers  $\ell$ ,*

$$|\{T \in \text{PYTab}(\lambda, \mu^n) : j^\ell(T) = T\}| = \pm \langle Q'_{1n}(e^{\frac{2\pi i \ell}{n}}) \circ s_\mu, s_\lambda \rangle.$$

*Remark 1.4.* The signs appearing in Theorems 1.1 and 1.2 are predictable, and depend upon  $\lambda$ ,  $d$ , and  $n$  only. Consult section 4, which contains the proofs of these theorems, for more details.

This article is by no means the first attempt at computing the coefficients in the expansions of power-sum plethysms. In 1995, Carré and Leclerc [5] devised a rule for splitting the square of a Schur function into its symmetric and antisymmetric parts, the crux of which was a demonstration that the coefficient  $\langle p_2 \circ s_\mu, s_\lambda \rangle$  counted, up to sign, the number of domino tableaux of shape  $\lambda$  and content  $\mu$  with Yamanouchi reading words. Two years later, in 1997, Lascoux, Leclerc, and Thibon [14] introduced a new family of symmetric functions, today referred to as LLT functions, and proposed that the plethysm  $p_{n/d}^d \circ s_\mu$  could be expressed as the specialization of an LLT function at an appropriate root of unity (as indeed  $p_{n/d}^d$  is the specialization of a Hall–Littlewood function). However, the Lascoux–Leclerc–Thibon conjecture remains unproven, and the Carré–Leclerc rule has not been generalized to cases beyond  $n = 2$ , for the concept of Yamanouchi reading words has not been extended to  $n$ -ribbon tableaux for  $n \geq 3$ .

Thus, Theorem 1.1 does not give the first combinatorial expression for the coefficient  $\langle p_2 \circ s_\mu, s_\lambda \rangle$ , but it distinguishes itself from the existing Carré–Leclerc formula by its natural compatibility with the Littlewood–Richardson rule, and it is sufficiently robust that the techniques involved in its derivation are applicable to a whole class of plethysm coefficients with  $n > 2$ , addressed in Theorem 1.2, which is new in content and in form. Furthermore, the author has shown in unpublished work that a bijection of Berenstein and Kirillov [2] between domino tableaux and tableaux stable under evacuation

restricts to a bijection between those tableaux specified in the Carré–Leclerc rule and in Theorem 1.1, respectively. It follows that Theorem 1.1 actually recovers the Carré–Leclerc result.

To prove Theorems 1.1 and 1.2, we turn to the theory of Lusztig canonical bases, which provides an algebraic setting for the Schützenberger actions evacuation and promotion. In particular, we consider an irreducible representation of  $GL_{mn}(\mathbb{C})$  for which there exists a basis indexed by the semistandard tableaux of shape  $\lambda$  with entries in  $\{1, 2, \dots, mn\}$  such that, if  $n = 2$ , the long element  $w_0 \in \mathfrak{S}_{mn} \hookrightarrow GL_{mn}$  permutes the basis elements (up to sign) by evacuation, and, if  $\lambda$  is rectangular, the long cycle  $c_{mn} \in \mathfrak{S}_{mn} \hookrightarrow GL_{mn}$  permutes the basis elements (up to sign) by promotion.

With a suitable basis in hand, we proceed to compute the character of the representation at a particular element of  $GL_{mn}$ . If  $n = 2$ , we compute

$$\chi(w_0 \cdot \text{diag}(x_1, x_2, \dots, x_m, x_m, \dots, x_2, x_1)),$$

and, if  $\lambda$  is rectangular, we compute

$$\chi(c_{mn}^{md} \cdot \text{diag}(y_1, y_2, \dots, y_d, y_1, y_2, \dots, y_d, \dots, y_1, y_2, \dots, y_d)),$$

where the block  $\text{diag}(y_1, y_2, \dots, y_d)$  occurs  $n/d$  times along the main diagonal, and  $y_i$  in turn represents the block  $\text{diag}(y_{i,1}, y_{i,2}, \dots, y_{i,m})$  for all  $1 \leq i \leq d$ .

These character evaluations pick out the fixed points of the relevant order  $n$  cyclic actions. Furthermore, they may be calculated by diagonalization of the indicated elements, for characters are class functions, and the values of the irreducible characters of  $GL_{mn}$  at diagonal matrices are well known. A careful inspection of the resulting formulae yields the desired identities.

This tactic is not original. The relationship between  $w_0$  and evacuation was first discovered by Berenstein and Zelevinsky [3] in 1996, in the context of a basis dual to Lusztig’s canonical basis. In this article, we opt for an essentially equivalent basis constructed by Skandera [24], which was used by Rhoades to detect the analogous relationship between  $c_{mn}$  and promotion. From the observations that  $w_0$  and  $c_{mn}$  lift the actions of evacuation and promotion, respectively, with respect to the dual canonical basis (or something like it), Stembridge [27] and Rhoades [20] deduced correspondences between fixed points of Schützenberger actions and ribbon tableaux, which inspired our results.

Recall that an  $r$ -ribbon tableau of shape  $\lambda$  is a tiling of the Young diagram of  $\lambda$  by connected skew diagrams with  $r$  boxes that contain no  $2 \times 2$  squares (referred to as  $r$ -ribbons), each labeled by a positive integer entry. (Thus, 1-ribbon tableaux are recognized as ordinary tableaux, and 2-ribbon tableaux are called domino tableaux.) If the entries of the  $r$ -ribbons are weakly increasing across each row and strictly increasing down each column, the  $r$ -ribbon tableau is called semistandard, by analogy with the definition of ordinary semistandard tableaux.

**Theorem 1.5** (Stembridge [27], Corollary 4.2). *Let  $\text{Tab}(\lambda, \overline{\mu}\mu)$  be the set of all semistandard tableaux of shape  $\lambda$  and content  $\overline{\mu}\mu$ , and let  $\xi$  act on  $\text{Tab}(\lambda, \overline{\mu}\mu)$  by the Schützenberger involution. Then*

$$|\{T \in \text{Tab}(\lambda, \overline{\mu}\mu) : \xi(T) = T\}|$$

*is the number of domino tableaux of shape  $\lambda$  and content  $\mu$ .*

**Theorem 1.6** (Rhoades [20], proof of Theorem 1.5). *Let  $\lambda$  be a rectangular partition, and let  $\text{Tab}(\lambda, \mu^n)$  be the set of all semistandard tableaux of shape  $\lambda$  and content  $\mu^n$ . Let  $j$  act on  $\text{Tab}(\lambda, \mu^n)$  by  $m$  iterations of jeu-de-taquin promotion. Then, for all positive integers  $d$  dividing  $n$ ,*

$$|\{T \in \text{Tab}(\lambda, \mu^n) : j^d(T) = T\}|$$

*is the number of  $(n/d)$ -ribbon tableaux of shape  $\lambda$  and content  $\mu^d$ .*

Unfortunately, the proofs of Theorems 1.5 and 1.6 cannot be directly adapted to obtain Theorems 1.1 and 1.2. In order for the Yamanouchi restrictions on our tableaux sets to be made to appear in our character evaluations, an additional point of subtlety is needed. We find relief in the insights offered us by the theory of Kashiwara crystals, which provides a framework not only for the study of the Schützenberger actions, but also for the reformulation of the Yamanouchi restrictions in terms of natural operators on semistandard tableaux.

Let  $\mathfrak{g}$  be a complex semisimple Lie algebra with weight lattice  $W$ , and choose a set of simple roots  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_t\}$ . A  $\mathfrak{g}$ -crystal is a finite set  $B$  equipped with a weight map  $\text{wt}: B \rightarrow W$  and a pair of raising and lowering operators  $e_i, f_i: B \rightarrow B \sqcup \{0\}$  for each  $i$  that obey certain conditions. Most notably, for all  $b \in B$ , if  $e_i \cdot b$  is nonzero, then  $\text{wt}(e_i \cdot b) = \text{wt}(b) + \alpha_i$ , and if  $f_i \cdot b$  is nonzero, then  $\text{wt}(f_i \cdot b) = \text{wt}(b) - \alpha_i$ .

If  $\mathfrak{g} = \mathfrak{sl}_{mn}$ , then  $W$  is a quotient of  $\mathbb{Z}^{mn}$ , and we may choose for our simple roots the images of the vectors  $\epsilon_i - \epsilon_{i+1}$  for  $1 \leq i \leq mn - 1$ , where  $\epsilon_i$  denotes the  $i^{\text{th}}$  standard basis vector for all  $1 \leq i \leq mn$ . In this case, we may take  $B$  to be the set of semistandard tableaux of shape  $\lambda$  with entries in  $\{1, 2, \dots, mn\}$ , with the weight of each tableau encoded in its content. As we see in section 3, there exists a suitable choice of operators  $e_i$  and  $f_i$  so that  $B$  assumes the structure of a  $\mathfrak{g}$ -crystal, and that the word of a tableau  $b \in B$  is Yamanouchi with respect to the letters  $i$  and  $i + 1$  if and only if  $e_i$  vanishes at  $b$ , and anti-Yamanouchi with respect to  $i$  and  $i + 1$  if and only if  $f_i$  vanishes at  $b$ . Furthermore, evacuation and promotion act on the set of crystal operators by conjugation (essentially), which implies that they act on the tableaux sets indicated in our main theorems.

We close the introduction with an outline of the rest of the article. In section 2, we provide the requisite background on tableaux and symmetric functions. After reviewing the rudimentary definitions, we introduce plethysms, and we end with the observation of Lascoux, Leclerc, and Thibon [14] that the classical relationship between tableaux and Schur functions

evinces a more general relationship between ribbon tableaux and power-sum plethysms of Schur functions. In section 3, we define Kashiwara crystals for a complex semisimple Lie algebra, before specializing to the  $\mathfrak{sl}_{mn}$  setting, where we show how to assign a crystal structure to the pertinent tableaux sets. We also define the Schützenberger actions and examine how they interact with the raising and lowering crystal operators. Both of these sections are expository.

Finally, in section 4, we present proofs of Theorems 1.1 and 1.2. Here the Berenstein–Zelevinsky [3] and Rhoades [20] lemmas underlying the proofs of Theorems 1.5 and 1.6 are fundamental.

## 2. TABLEAUX AND SYMMETRIC FUNCTION BACKGROUND

In this section, we discuss the basic facts about Young tableaux and symmetric functions that are necessary for this article to be understood and placed in its proper context.<sup>1</sup> For the sake of completeness, we begin with the definition of a composition and that of a partition.

**Definition 2.1.** A *composition*  $\kappa$  of a positive integer  $k$  is a sequence of nonnegative integers  $(\kappa_1, \kappa_2, \dots, \kappa_s)$  satisfying  $\kappa_1 + \kappa_2 + \dots + \kappa_s = k$ . We refer to  $s$  as the *number of parts* of  $\kappa$ , and we write  $|\kappa| = k$ .

**Definition 2.2.** A composition  $\kappa$  of a positive integer  $k$  is a *partition* of  $k$  if  $\kappa$  is weakly decreasing. In this case, we write  $\kappa \vdash k$ .

*Remark 2.3.* We adopt the standard convention of identifying compositions (or partitions) that differ only by terminal zeroes, *except* when we require the number of parts of a composition to be well-defined. For instance, in defining the tableaux sets from the introduction, we consider  $\mu$  and  $\lambda$  to have  $m$  and  $mn$  parts, respectively, for an arbitrary  $m$  such that  $\mu$  has at most  $m$  positive parts and  $\lambda$  has at most  $mn$  positive parts, *and then we consider  $m$  to be fixed*. It should be clear that the choice of  $m$  does *not* affect the cardinalities of the tableaux sets in question.

There is a natural action of  $\mathfrak{S}_s$  on the set of compositions with  $s$  parts given by  $w \cdot (\kappa_1, \kappa_2, \dots, \kappa_s) = (\kappa_{w^{-1}(1)}, \kappa_{w^{-1}(2)}, \dots, \kappa_{w^{-1}(s)})$  for all permutations  $w$ . This is important later.

To each partition, we may associate a pictorial representation known as a Young diagram. This association allows for the introduction of the basic combinatorial object of this article — the Young tableau.

**Definition 2.4.** Let  $k$  be a positive integer, and let  $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_s)$  be a partition of  $k$ . A *Young diagram* of *shape*  $\kappa$  is a collection of unit squares (referred to as boxes) arranged in left-justified rows, such that the number of boxes in the  $i^{\text{th}}$  row is  $\kappa_i$  for all  $1 \leq i \leq s$ .

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<sup>1</sup>More comprehensive accounts of the fundamentals can be found in Stanley [25], Chapter 7 or Fulton [7], Chapters 1-6 (of the two treatments, Fulton’s is the more leisurely). For more details on plethysms, a reference *par excellence* is MacDonald [18], Chapter 1, but the presentation is considerably more abstract.

**Definition 2.5.** Let  $\iota$  and  $\kappa$  be partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ . The set-theoretic difference of a Young diagram of shape  $\kappa$  and a Young diagram of shape  $\iota$  is a *skew diagram* of shape  $\kappa/\iota$ . We write  $|\kappa/\iota|$  for the total area of the skew diagram,  $|\kappa| - |\iota|$ .

**Definition 2.6.** A *path* in a skew diagram is a sequence of squares such that every pair of consecutive squares in the path shares a side in the skew diagram. A skew diagram is *connected* if, for any two squares in the skew diagram, there exists a path beginning at one square and ending at the other.

**Definition 2.7.** An *r-ribbon* is a connected skew diagram of total area  $r$  that contains no  $2 \times 2$  block of squares.

**Definition 2.8.** Let  $\kappa$  be a partition, and let *r-ribbons* be successively removed from a Young diagram of  $\kappa$  such that the figure remains a Young diagram after each *r-ribbon* is removed. When no further *r-ribbons* can be removed, the partition corresponding to the resulting Young diagram is the *r-core* of  $\kappa$ .

**Definition 2.9.** Let  $\kappa$  be partition, and suppose that the *r-core* of  $\kappa$  is empty. An *r-ribbon diagram* of shape  $\kappa$  is a tiling of a Young diagram of shape  $\kappa$  by *r-ribbons*.

**Definition 2.10.** Let  $\kappa$  be a partition of  $k$ , and let  $\eta = (\eta_1, \eta_2, \dots, \eta_t)$  be a composition of  $k$ . A *semistandard Young tableau* of shape  $\kappa$  and content  $\eta$  is a filling of a Young diagram of shape  $\kappa$  by positive integer entries, with one entry in each box, such that the entries are weakly increasing across each row and strictly increasing down each column, and such that the integer  $i$  appears as an entry  $\eta_i$  times for all  $1 \leq i \leq t$ . A semistandard tableau of shape  $\kappa$  and content  $\eta$  is *standard* if  $\eta_i = 1$  for all  $1 \leq i \leq k$ .

**Definition 2.11.** Let  $\iota$  and  $\kappa$  be partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ . Let  $\eta = (\eta_1, \eta_2, \dots, \eta_t)$  be a composition of  $|\kappa/\iota|$ . A *semistandard skew tableau* of shape  $\kappa/\iota$  and content  $\eta$  is a filling of a skew diagram of shape  $\kappa/\iota$  by positive integer entries, with one entry in each box, such that the entries are weakly increasing across each row and strictly increasing down each column, and such that the integer  $i$  appears as an entry  $\eta_i$  times for all  $1 \leq i \leq t$ .

**Definition 2.12.** Let  $\kappa$  be a partition of  $k$ , and suppose that the *r-core* of  $\kappa$  is empty. Let  $\eta = (\eta_1, \eta_2, \dots, \eta_t)$  be a composition of  $\frac{k}{r}$ . A *semistandard r-ribbon tableau* of shape  $\kappa$  and content  $\eta$  is a filling of an *r-ribbon diagram* of shape  $\kappa$  by positive integer entries, with one entry in each *r-ribbon*, such that the entries are weakly increasing across each row and strictly increasing down each column, and such that the integer  $i$  appears as an entry  $\eta_i$  times for all  $1 \leq i \leq t$ .

To each semistandard tableau, we may associate a word that contains all the entries of the tableau, called the reading word. It turns out that the

tableau may be uniquely recovered from its reading word via the Schensted insertion algorithm (cf. Fulton [7], Chapters 1-3), with which we are not herein concerned.

**Definition 2.13.** Given a semistandard tableau  $T$ , the *reading word* of  $T$ , which we denote by  $w(T)$ , is the word obtained by reading the entries of  $T$  from bottom to top in each column, beginning with the leftmost column, and ending with the rightmost column.

If  $T$  is a tableau of shape  $\kappa \vdash k$  and content  $\eta = (\eta_1, \eta_2, \dots, \eta_t)$ , then  $w(T)$  is a word of length  $k$  on the alphabet  $\{1, 2, \dots, t\}$ , and the integer  $i$  appears as a letter  $\eta_i$  times for all  $1 \leq i \leq t$ .

To date, it has not been determined whether reading words can be systematically assigned to semistandard  $r$ -ribbon tableaux for  $r \geq 3$ . However, if  $r = 2$ , the notion of reading word remains viable. We refer to 2-ribbon diagrams as *domino diagrams* and to 2-ribbon tableaux as *domino tableaux*.

**Definition 2.14.** Given a semistandard domino tableau  $T$ , the *reading word* of  $T$ , which we denote by  $w(T)$ , is the word obtained by reading the entries of  $T$  from bottom to top in each column, beginning with the leftmost column, and ending with the rightmost column, such that the entry in each horizontal domino is read only once — in the column in which the domino is first encountered.

We now introduce the properties that characterize the reading words of the tableaux specified in our main theorems, as well as in the Littlewood–Richardson rule.

**Definition 2.15.** A word  $w = w_1 w_2 \cdots w_k$  on the alphabet  $\{1, 2, \dots, t\}$  is *Yamanouchi* (*anti-Yamanouchi*) with respect to the integers  $i$  and  $i + 1$  if, when it is read backwards from the end to any letter, the resulting sequence  $w_k, w_{k-1}, \dots, w_j$  contains at least (at most) as many instances of  $i$  as of  $i + 1$ .

**Definition 2.16.** A word  $w$  on the alphabet  $\{1, 2, \dots, t\}$  is *Yamanouchi* (*anti-Yamanouchi*) in the subset  $\{i, i + 1, \dots, i'\}$  if it is Yamanouchi (*anti-Yamanouchi*) with respect to each pair of consecutive integers in  $\{i, i + 1, \dots, i'\}$ .

That concludes our litany of combinatorial definitions. We turn to a brief overview of symmetric polynomials and symmetric functions.

Let  $\Lambda_m$  be the ring of symmetric polynomials in  $m$  variables; let  $\Lambda$  be the ring of symmetric functions, and let  $\Lambda_m^k$  and  $\Lambda^k$  be their respective degree  $k$  homogeneous components.

**Definition 2.17.** Let  $\kappa$  be a partition of a positive integer  $k$ . For all compositions  $\eta$  of  $k$ , we denote the monomial  $x_1^{\eta_1} x_2^{\eta_2} \cdots$  by  $x^\eta$ , and, for all tableaux  $T$  of shape  $\kappa$  and content  $\eta$ , we write  $x^T$  for  $x^\eta$ . The *Schur function* associated to  $\kappa$  in the variables  $x_1, x_2, \dots$  is  $s_\kappa := \sum_T x^T$ , where the sum ranges over all tableaux  $T$  of shape  $\kappa$ . For all  $m$ , the *Schur polynomial* associated to  $\kappa$  in the  $m$  variables  $x_1, x_2, \dots, x_m$  is  $s_\kappa(x_1, x_2, \dots, x_m)$ .



It is well known that the Schur polynomials in  $m$  variables associated to partitions of  $k$  with at most  $m$  positive parts form a basis for the  $\mathbb{Z}$ -module  $\Lambda_m^k$  (cf. Proposition 1 in Fulton [7], Chapter 6). Similarly, the Schur functions associated to partitions of  $k$  form a basis for the  $\mathbb{Z}$ -module  $\Lambda^k$ . Since  $\Lambda_m = \bigoplus_{k \geq 0} \Lambda_m^k$  and  $\Lambda = \bigoplus_{k \geq 0} \Lambda^k$ , it follows that the Schur polynomials in  $m$  variables associated to partitions with at most  $m$  positive parts form a basis for  $\Lambda_m$ , and that the Schur functions form a basis for  $\Lambda$ . We now define an inner product on  $\Lambda$  by decreeing that the Schur basis be orthonormal.

**Definition 2.18.** Let  $\langle \cdot, \cdot \rangle: \Lambda \times \Lambda \rightarrow \mathbb{Z}$  be an inner product given by  $\langle s_\iota, s_\kappa \rangle = \delta_{\iota, \kappa}$  for all partitions  $\iota$  and  $\kappa$ , where  $\delta_{\iota, \kappa}$  denotes the Kronecker delta.

Thus, if  $f$  is a symmetric function, there exists a unique expression for  $f$  as a linear combination of Schur functions, and the coefficients are given by the inner product:  $f = \sum_{\kappa} \langle f, s_{\kappa} \rangle s_{\kappa}$ . We refer to the sum as the *expansion* of  $f$  on the Schur basis, and to the inner products  $\langle f, s_{\kappa} \rangle$  as the *expansion coefficients*. The Littlewood–Richardson rule is a combinatorial description for the expansion coefficients of a product of a Schur functions.

**Theorem 2.19** (Littlewood–Richardson). *Let  $\iota$  and  $\kappa$  be partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ . Let  $\theta$  be a partition of  $|\kappa/\iota|$ . Then  $\langle s_{\iota} s_{\theta}, s_{\kappa} \rangle$  counts the number of semistandard skew tableaux of shape  $\kappa/\iota$  and content  $\theta$  with Yamanouchi reading word.*

Fortuitously, the Littlewood–Richardson coefficients also appear in the expansion of a single Schur polynomial in multiple variable sets as a sum of products of Schur polynomials in each constituent variable set.

**Theorem 2.20.** *Let  $d$  be a positive integer, and let*

$$\{a_{1,1}, a_{1,2}, \dots, a_{1,m_1}\}, \{a_{2,1}, a_{2,2}, \dots, a_{2,m_2}\}, \dots, \{a_{d,1}, a_{d,2}, \dots, a_{d,m_d}\}$$

*be a collection of  $d$  variable sets denoted by  $a_1, a_2, \dots, a_d$ , respectively. Then*

$$s_{\kappa}(a_1, a_2, \dots, a_d) = \sum_{\theta_1, \theta_2, \dots, \theta_d} \langle s_{\kappa}, s_{\theta_1} s_{\theta_2} \cdots s_{\theta_d} \rangle s_{\theta_1}(a_1) s_{\theta_2}(a_2) \cdots s_{\theta_d}(a_d),$$

*where  $\theta_1, \theta_2, \dots, \theta_d$  ranges over all  $d$ -tuples of partitions.*

*Proof.* The proof is by induction on  $d$ . The base case  $d = 2$  is proven in Chapter 7, Section 15 of Stanley [25]. The inductive step is handled identically.  $\square$

Since the Schur polynomials form a basis for the ring of symmetric polynomials, it follows that all symmetric polynomials in multiple variable sets may be expanded as a sum of products of Schur polynomials in each constituent variable set, with the expansion coefficients given by symmetric function inner products. The expansion coefficients of this latter kind are also uniquely determined.

**Corollary 2.21.** *Let  $d$  be a positive integer, and let*

$$\{a_{1,1}, a_{1,2}, \dots, a_{1,m_1}\}, \{a_{2,1}, a_{2,2}, \dots, a_{2,m_2}\}, \dots, \{a_{d,1}, a_{d,2}, \dots, a_{d,m_d}\}$$

*be a collection of  $d$  variable sets denoted by  $a_1, a_2, \dots, a_d$ , respectively. For all  $f \in \Lambda$ ,*

$$f(a_1, a_2, \dots, a_d) = \sum_{\theta_1, \theta_2, \dots, \theta_d} \langle f, s_{\theta_1} s_{\theta_2} \cdots s_{\theta_d} \rangle s_{\theta_1}(a_1) s_{\theta_2}(a_2) \cdots s_{\theta_d}(a_d),$$

*where  $\theta_1, \theta_2, \dots, \theta_d$  ranges over all  $d$ -tuples of partitions.*

**Corollary 2.22.** *Let  $d$  be a positive integer, and let*

$$\{a_{1,1}, a_{1,2}, \dots, a_{1,m_1}\}, \{a_{2,1}, a_{2,2}, \dots, a_{2,m_2}\}, \dots, \{a_{d,1}, a_{d,2}, \dots, a_{d,m_d}\}$$

*be a collection of  $d$  variable sets denoted by  $a_1, a_2, \dots, a_d$ , respectively. Then the set of products  $\{s_{\theta_1}(a_1) s_{\theta_2}(a_2) \cdots s_{\theta_d}(a_d)\}_{\theta_1, \theta_2, \dots, \theta_d}$ , where  $\theta_1, \theta_2, \dots, \theta_d$  ranges over all  $d$ -tuples of partitions, constitutes a basis for the ring of symmetric polynomials in the variable sets  $a_1, a_2, \dots, a_d$ .*

Finally, we come to the definition of plethysm, taken from Macdonald [18].

**Definition 2.23.** Let  $f, g \in \Lambda$ , and let  $g$  be written as a sum of monomials, so that  $g = \sum_{\eta} u_{\eta} x^{\eta}$ , where  $\eta$  ranges over an infinite set of compositions. Let  $\{y_i\}_{i=1}^{\infty}$  be a collection of proxy variables defined by  $\prod_{i=1}^{\infty} (1 + y_i t) = \prod_{\eta} (1 + x^{\eta} t)^{u_{\eta}}$ . The *plethysm* of  $f$  and  $g$ , which we denote by  $f \circ g$ , is the symmetric function  $f(y_1, y_2, \dots)$ .

*Remark 2.24.* Although the relation  $\prod_{i=1}^{\infty} (1 + y_i t) = \prod_{\eta} (1 + x^{\eta} t)^{u_{\eta}}$  only determines the elementary symmetric functions in the variables  $y_1, y_2, \dots$  (viz.,  $e_1(y_1, y_2, \dots) = y_1 + y_2 + \cdots$ ,  $e_2(y_1, y_2, \dots) = y_1 y_2 + y_1 y_3 + y_2 y_3 + \cdots$ , etc.), it is well known that the ring of symmetric functions is generated as a  $\mathbb{Z}$ -algebra by the elementary symmetric functions, so the plethysm  $f \circ g = f(y_1, y_2, \dots)$  is indeed well-defined.

The following observation follows immediately from Definition 2.23.

**Proposition 2.25.** *For all  $f \in \Lambda$ , the map  $\Lambda \rightarrow \Lambda$  given by  $g \mapsto g \circ f$  is a ring homomorphism.*

There exists a family of symmetric functions for which the other choice of map given by plethysm, i.e.  $g \mapsto f \circ g$ , is also a ring homomorphism, for all  $f$  belonging to this family.

**Definition 2.26.** For all positive integers  $k$ , the  $k^{\text{th}}$  *power-sum symmetric function* in the variables  $x_1, x_2, \dots$  is  $p_k := x_1^k + x_2^k + \cdots$ .

**Proposition 2.27.** *Let  $g \in \Lambda$ , and let  $k$  be a positive integer. Then  $p_k \circ g = g \circ p_k = g(x_1^k, x_2^k, \dots)$ .*

*Proof.* As in Definition 2.23, say that  $g = \sum_{\eta} u_{\eta} x^{\eta}$ . Taking logarithms of each side in the equality  $\prod_{i=1}^{\infty} (1 + y_i t) = \prod_{\eta} (1 + x^{\eta} t)^{u_{\eta}}$ , we obtain

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} y_i^k t^k = \sum_{\eta} \left( u_{\eta} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x^{\eta})^k t^k \right).$$

Interchanging the order of summation on each side yields  $p_k(y_1, y_2, \dots) = \sum_{\eta} u_{\eta} (x^{\eta})^k = g(x_1^k, x_2^k, \dots)$ . Since  $p_k \circ g = p_k(y_1, y_2, \dots)$ , it follows that  $p_k \circ g = g(x_1^k, x_2^k, \dots)$ . It should be clear that  $g \circ p_k = g(x_1^k, x_2^k, \dots)$  as well.  $\square$

We may conclude that the map given by  $g \mapsto p_k \circ g$  is a ring homomorphism for all positive integers  $k$ . We are therefore permitted to introduce an adjoint operator, which we denote by  $\varphi_k$ , given by  $f \mapsto \sum_{\kappa} \langle f, p_k \circ s_{\kappa} \rangle s_{\kappa}$ , where the sum ranges over all partitions  $\kappa$ . Clearly, the equality  $\langle \varphi_k(f), g \rangle = \langle f, p_k \circ g \rangle$  holds for all  $f, g \in \Lambda$ , which explains the nomenclature.

Let  $\kappa$  be a partition. Just as the ordinary tableaux of shape  $\kappa$  index the monomials of the Schur function  $s_{\kappa}$ , the  $k$ -ribbon tableaux of shape  $\kappa$  index the monomials of the symmetric function  $\varphi_k(s_{\kappa})$ .

**Theorem 2.28.** *Let  $\kappa$  be a partition, and suppose that the  $k$ -core of  $\kappa$  is empty. For all compositions  $\eta$  of  $\frac{|\kappa|}{k}$ , we denote the monomial  $x_1^{\eta_1} x_2^{\eta_2} \dots$  by  $x^{\eta}$ , and, for all  $k$ -ribbon tableaux  $T$  of shape  $\kappa$  and content  $\eta$ , we write  $x^T$  for  $x^{\eta}$ . Then  $\varphi_k(s_{\kappa}) = \epsilon_k(\kappa) \sum_T x^T$ , where the sum ranges over all  $k$ -ribbon tableaux of shape  $\kappa$ , and  $\epsilon_k(\kappa)$  denotes the  $k$ -sign of  $\kappa$ .*

*Proof.* Let  $(\kappa^{(1)}, \kappa^{(2)}, \dots, \kappa^{(k)})$  be the  $k$ -quotient of  $\kappa$  (we omit a formal definition of  $k$ -quotient for fear it would take us too far afield, but one can be found in James–Kerber [9]). Since the  $k$ -core of  $\kappa$  is empty, it follows from a result of Littlewood [16] that  $\varphi_k(s_{\kappa}) = \epsilon_k(\kappa) s_{\kappa^{(1)}} s_{\kappa^{(2)}} \dots s_{\kappa^{(k)}}$ . However, from Equation 24 in Lascoux–Leclerc–Thibon [14], we see that  $s_{\kappa^{(1)}} s_{\kappa^{(2)}} \dots s_{\kappa^{(k)}} = \sum_T x^T$ , where the sum ranges over all  $k$ -ribbon tableaux of shape  $\kappa$ , as desired. (This identity is an algebraic restatement of a bijection between  $k$ -tuples of tableaux of shapes  $(\kappa^{(1)}, \kappa^{(2)}, \dots, \kappa^{(k)})$  and  $k$ -ribbon tableaux of shape  $\kappa$ , due in its original form to Stanton and White [26].)  $\square$

In view of Theorem 2.28, it is natural to ask if there is an analogue of the Littlewood–Richardson rule that describes the expansion coefficients of the power-sum plethysms  $p_n \circ s_{\mu}$ , or, more generally,  $p_{n/d}^d \circ s_{\mu} = (p_{n/d} \circ s_{\mu})^d$ , for  $d$  dividing  $n$ . In the following sections, we see how this article provides a partial affirmative answer.

### 3. CRYSTAL STRUCTURE ON TABLEAUX

For a complex semisimple Lie algebra  $\mathfrak{g}$ , Kashiwara’s  $\mathfrak{g}$ -crystals constitute a class of combinatorial models patterned on representations of  $\mathfrak{g}$ . If  $\mathfrak{g}$  is simply laced, there exists a set of axioms, enumerated by Stembridge [28],

that characterize the crystals arising directly from  $\mathfrak{g}$ -representations, which he calls regular. Given a partition  $\kappa$  with  $s$  parts, the combinatorics of the weight space decomposition of the irreducible  $\mathfrak{sl}_s$ -representation with highest weight encoded in  $\kappa$  is captured in the regular  $\mathfrak{sl}_s$ -crystal structure assigned to the semistandard tableaux of shape  $\kappa$  with entries in  $\{1, 2, \dots, s\}$ .

In this section, we review the crystal structure on tableaux, and we observe that it offers a natural setting for the consideration of evacuation and promotion, due to the relationship between these actions and the raising and lowering crystal operators. We also see that the crystal perspective facilitates a recasting of the Yamanouchi conditions on tableaux reading words in terms of the vanishing or nonvanishing of the raising and lowering operators at the corresponding tableaux, viewed as crystal elements. We end with restatements of the Littlewood–Richardson rule that validate our claim that Theorems 1.1 and 1.2 are indeed analogous to it.

To provide a thorough treatment of the background,<sup>2</sup> we begin with the definition of a crystal, following Joseph [10], and that of a regular crystal, following Stembridge [28].

**Definition 3.1.** Let  $\mathfrak{g}$  be a complex semisimple Lie algebra with weight lattice  $W$ . Let  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_t\}$  be a choice of simple roots, and let  $\{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_t^\vee\}$  be the corresponding simple coroots. A  $\mathfrak{g}$ -crystal is a finite set  $B$  equipped with a map  $\text{wt}: B \rightarrow W$  and a pair of operators  $e_i, f_i: B \rightarrow B \sqcup \{0\}$  for each  $1 \leq i \leq t$  that satisfy the following conditions:

- (i)  $\max\{\ell: f_i^\ell \cdot b \neq 0\} - \max\{\ell: e_i^\ell \cdot b \neq 0\} = \langle \text{wt}(b), \alpha_i^\vee \rangle$  for all  $b \in B$ ;
- (ii)  $e_i \cdot b \neq 0$  implies  $\text{wt}(e_i \cdot b) = \text{wt}(b) + \alpha_i$  and  $f_i \cdot b \neq 0$  implies  $\text{wt}(f_i \cdot b) = \text{wt}(b) - \alpha_i$  for all  $b \in B$ ;
- (iii)  $b' = e_i \cdot b$  if and only if  $b = f_i \cdot b'$  for all  $b, b' \in B$ .

We refer to  $e_i$  as the *raising operator* associated to  $\alpha_i$ , and we refer to  $f_i$  as the *lowering operator* associated to  $\alpha_i$ . We write  $\epsilon_i(b) := \max\{\ell: e_i^\ell \cdot b \neq 0\}$  for the maximum number of times the raising operator  $e_i$  may be applied to  $b$  without vanishing, and we write  $\phi_i(b) := \max\{\ell: f_i^\ell \cdot b \neq 0\}$  for the maximum number of times the lowering operator  $f_i$  may be applied to  $b$  without vanishing. We also define, for all  $1 \leq i, j \leq t$ :

- $\Delta_i \epsilon_j(b) := \epsilon_j(b) - \epsilon_j(e_i b)$ ;
- $\Delta_i \phi_j(b) := \phi_j(e_i b) - \phi_j(b)$ ;
- $\nabla_i \epsilon_j(b) := \epsilon_j(f_i b) - \epsilon_j(b)$ ;
- $\nabla_i \phi_j(b) := \phi_j(b) - \phi_j(f_i b)$ .

**Definition 3.2.** Let  $B$  and  $B'$  be  $\mathfrak{g}$ -crystals. A map of sets  $\pi: B \rightarrow B'$  is a *morphism of crystals* if  $\pi e_i = e_i \pi$  and  $\pi f_i = f_i \pi$  for all  $1 \leq i \leq t$ . (Here we tacitly stipulate  $\pi(0) := 0$ .) If  $\pi$  is bijective, we say  $\pi$  is an *isomorphism*.

<sup>2</sup>For more details on crystals, consult Joseph [10]. For more on the crystal structure on tableaux, see Kashiwara–Nakashima [11]. For more on *jeu de taquin*, see Fulton [7]. For more on evacuation and *jeu-de-taquin* promotion, see Schützenberger [22] and Shimozono [23]. For more on promotion in crystals, see Bandlow–Schilling–Thiéry [1].

**Definition 3.3.** A  $\mathfrak{g}$ -crystal  $B$  is *regular* if the following conditions hold for all  $i \neq j$  and  $b \in B$ .

- (iv)  $\Delta_i \epsilon_j(b), \Delta_i \phi_j(b), \nabla_i \epsilon_j(b), \nabla_i \phi_j(b) \leq 0$ .
- (va) If  $e_i b, e_j b \neq 0$  and  $\Delta_i \epsilon_j(b) = 0$ , then  $e_i e_j b = e_j e_i b$  and  $\nabla_j \phi_i(e_i e_j b) = 0$ .
- (vb) If  $f_i b, f_j b \neq 0$  and  $\nabla_i \phi_j(b) = 0$ , then  $f_i f_j b = f_j f_i b$  and  $\Delta_j \epsilon_i(f_i f_j b) = 0$ .
- (via) If  $e_i b, e_j b \neq 0$  and  $\Delta_i \epsilon_j(b) = \Delta_j \epsilon_i(b) = -1$ , then  $e_i e_j^2 e_i b = e_j e_i^2 e_j b$  and  $\nabla_i \phi_j(e_i e_j^2 e_i b) = \nabla_j \phi_i(e_i e_j^2 e_i b) = -1$ .
- (vib) If  $f_i b, f_j b \neq 0$  and  $\nabla_i \phi_j(b) = \nabla_j \phi_i(b) = -1$ , then  $f_i f_j^2 f_i b = f_j f_i^2 f_j b$  and  $\Delta_i \epsilon_j(f_i f_j^2 f_i b) = \Delta_j \epsilon_i(f_i f_j^2 f_i b) = -1$ .

**Definition 3.4.** A  $\mathfrak{g}$ -crystal  $B$  is *connected* if the underlying graph, in which elements  $b$  and  $b'$  are joined by an edge if there exists  $i$  such that  $e_i \cdot b = b'$  or  $e_i \cdot b' = b$ , is connected. We refer to the connected components of the underlying graph as the *connected components* of  $B$ .

*Remark 3.5.* Regular connected  $\mathfrak{g}$ -crystals should be thought of in analogy with irreducible representations of  $\mathfrak{g}$ .

**Definition 3.6.** Let  $B$  be a  $\mathfrak{g}$ -crystal. An element  $b \in B$  is a *highest weight element* if  $e_i$  vanishes at  $b$  for all  $i$ . If  $b$  is the unique highest weight element of  $B$ , then  $B$  is a *highest weight crystal* of highest weight  $\text{wt}(b)$ .

This terminology is compatible with the natural partial order on  $B$  given by the restriction of the root order on  $W$  to the image of  $\text{wt}$  in the sense that, if  $B$  is connected, the maximal elements under this partial order coincide precisely with the highest weight elements of  $B$ .

Every highest weight crystal is connected, but the converse does not hold in general. However, if we restrict our attention to regular crystals, then saying a crystal is connected is equivalent to saying it is a highest weight crystal. Furthermore, a regular connected crystal  $B$  with highest weight  $b$  is uniquely determined by the values  $\phi_i(b)$  for  $1 \leq i \leq t$ .

**Proposition 3.7** (Stembridge [28]). *Let  $\mathfrak{g}$  be simply laced, and let  $B$  be a regular connected  $\mathfrak{g}$ -crystal. Then  $B$  is a highest weight crystal.*

**Proposition 3.8** (Stembridge [28]). *Let  $B$  and  $B'$  be regular connected  $\mathfrak{g}$ -crystals with highest weight elements  $b$  and  $b'$ , respectively. If  $\phi_i(b) = \phi_i(b')$  for all  $1 \leq i \leq t$ , then  $B$  and  $B'$  are isomorphic.*

Specializing to the case  $\mathfrak{g} = \mathfrak{sl}_s$ , we take as our Cartan subalgebra  $\mathfrak{h}$  the subspace of traceless diagonal matrices, and we identify  $\mathfrak{h}^*$  with the quotient space  $\mathbb{C}^s / (\epsilon_1 + \epsilon_2 + \dots + \epsilon_s)\mathbb{C}$ , where  $\epsilon_i$  denotes the  $i^{\text{th}}$  standard basis vector for all  $1 \leq i \leq s$ . Writing  $E_i$  for the image of  $\epsilon_i$  in  $\mathfrak{h}^*$  for all  $i$ , we note that the weight lattice  $W$  is generated over  $\mathbb{Z}$  by  $\{E_1, E_2, \dots, E_s\}$ , and we choose the set of simple roots  $\{\alpha_1, \alpha_2, \dots, \alpha_{s-1}\}$  in accordance with the rule  $\alpha_i := E_i - E_{i+1}$  for all  $1 \leq i \leq s-1$ .

To each partition  $\kappa$  with at most  $s$  positive parts, we consider  $\kappa$  as a partition with  $s$  parts, and we impose an  $\mathfrak{sl}_s$ -crystal structure on the tableaux of shape  $\kappa$  with entries in  $\{1, 2, \dots, s\}$  such that the highest weight is  $\kappa + (\epsilon_1 + \epsilon_2 + \dots + \epsilon_s)\mathbb{Z}$ . To do so, we begin by defining an  $\mathfrak{sl}_s$ -crystal structure on the skew tableaux of shape  $\kappa/\iota$ , and then we reduce to the case in which the partition  $\iota$  is empty.

**Proposition 3.9** (Kashiwara–Nakashima [11]). *Let  $\kappa$  and  $\iota$  be partitions, each with  $s$  parts, such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ . Let  $B_{\kappa/\iota}$  be the set of semistandard skew tableaux of shape  $\kappa/\iota$  with entries in  $\{1, 2, \dots, s\}$ .*

*Let the maps*

$$\begin{aligned} \text{wt}: B_{\kappa/\iota} &\rightarrow \mathbb{Z}^s / (\epsilon_1 + \epsilon_2 + \dots + \epsilon_s)\mathbb{Z} \\ h_{i,j}, k_{i,j}: B_{\kappa/\iota} &\rightarrow \mathbb{Z} \\ e_i, f_i: B_{\kappa/\iota} &\rightarrow B_{\kappa/\iota} \sqcup \{0\} \end{aligned}$$

*be given for all  $1 \leq i \leq s-1$  and  $j \in \mathbb{N}$  by stipulating, for all  $T \in B_{\kappa/\iota}$ :*

- $\text{wt}(T)$  to be the image in  $\mathbb{Z}^s / (\epsilon_1 + \epsilon_2 + \dots + \epsilon_s)\mathbb{Z}$  of the content of  $T$ ;
- $h_{i,j}(T)$  to be the number of occurrences of  $i+1$  in the  $j^{\text{th}}$  column of  $T$  or to the right minus the number of occurrences of  $i$  in the  $j^{\text{th}}$  column of  $T$  or to the right;
- $k_{i,j}(T)$  to be the number of occurrences of  $i$  in the  $j^{\text{th}}$  column of  $T$  or to the left minus the number of occurrences of  $i+1$  in the  $j^{\text{th}}$  column of  $T$  or to the left;
- $e_i(T)$  to be the skew tableau with an  $i$  in place of an  $i+1$  in the rightmost column for which  $h_{i,j}(T)$  is maximal and positive if such a column exists, and 0 otherwise;
- $f_i(T)$  to be the skew tableau with an  $i+1$  in place of an  $i$  in the leftmost column for which  $k_{i,j}(T)$  is maximal and positive if such a column exists, and 0 otherwise.

*Then the set  $B_{\kappa/\iota}$  equipped with the map  $\text{wt}$  and the operators  $e_i, f_i$  for all  $1 \leq i \leq s-1$  is an  $\mathfrak{sl}_s$ -crystal.*

**Proposition 3.10** (Stembridge [28]). *Let  $\kappa$  be a partition with  $s$  parts. The  $\mathfrak{sl}_s$ -crystal  $B_\kappa := B_{\kappa/\emptyset}$  defined in Proposition 3.9 is a regular connected crystal of highest weight  $\kappa + (\epsilon_1 + \epsilon_2 + \dots + \epsilon_s)\mathbb{Z}$ . The highest weight element is the unique tableau of shape  $\kappa$  and content  $\kappa$ .*

Fundamental to the study of skew tableaux is a procedure devised by Schützenberger for transforming skew tableaux into tableaux of left-justified shape. Given a skew tableau  $T$  of shape  $\kappa/\iota$ , *jeu de taquin* calls for the boxes in the Young diagram of shape  $\iota$  to be relocated one at a time from the northwest to the southeast of  $T$  via a sequence of successive slides. Because

*jeu-de-taquin* slides commute with the raising and lowering operators, we consider *jeu de taquin* to respect the crystal structure on tableaux. Thus, the observation that  $B_{\kappa/\iota}$  may be realized as a crystal even in cases in which  $\iota$  is nonempty is less superfluous than it may seem. (In fact, it turns out that  $B_{\kappa/\iota}$  is a regular crystal regardless of whether  $\iota$  is empty or not, but, when  $\iota$  is nonempty, the assertion that it is a crystal is sufficient for our purposes here.)

**Definition 3.11.** Let  $\kappa$  and  $\iota$  be partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ . In a skew diagram of shape  $\kappa/\iota$ , an *inside corner* (*outside corner*) is a box in the Young diagram of shape  $\iota$  ( $\kappa$ ) for which neither the box below nor the box to the right are in  $\iota$  ( $\kappa$ ).

**Definition 3.12.** Let  $\kappa$  and  $\iota$  be nonempty partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ , and let  $T$  be a semistandard skew tableau of shape  $\kappa/\iota$ . A *jeu-de-taquin slide* on  $T$  consists of the following steps.

- (i) An inside corner of  $\kappa/\iota$  is designated the “empty box.”
- (ii) The entry in  $T$  in the box below the empty box is compared to the entry in the box to the right of the empty box.
- (iii) If the entry in the box below is less than or equal to the entry in the box to the right (or if the box to the right is not in  $\kappa$ ), then the entry in the box below is “slid” into the empty box, which is no longer considered empty, and the box newly free of an entry is designated the “empty box.”
- (iv) If the entry in the box below is greater than the entry in the box to the right (or if the box below is not in  $\kappa$ ), then the entry in the box to the right is “slid” into the empty box, which is no longer considered empty, and the box newly free of an entry is designated the “empty box.”
- (v) The process specified in steps (ii)-(iv) is iterated until an outside corner of  $\kappa/\iota$  is designated the “empty box,” at which point the empty box is removed from the diagram.

**Definition 3.13.** Let  $\kappa$  and  $\iota$  be partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ , and let  $T$  be a semistandard skew tableau of shape  $\kappa/\iota$ . Schützenberger’s *jeu de taquin* is the procedure under which *jeu-de-taquin* slides are performed on  $T$  until the remaining diagram is left-justified. We refer to the resulting tableau of left-justified shape as the *rectification* of  $T$ , which we denote by  $\text{Rect}(T)$ .

*Remark 3.14.* To see that *jeu de taquin* is well-defined, observe that the rectification of a tableau is independent of the sequence of choices of inside corners via which it is obtained (cf. Fulton [7], Chapter 2, Corollary 1).

**Proposition 3.15** (Lascoux–Schützenberger [15]). *Let  $\kappa$  and  $\iota$  be nonempty partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ , and let  $C$  be an inside corner of  $\kappa/\iota$ . For all semistandard skew tableaux  $T$  of shape  $\kappa/\iota$ , let*

$\text{jdt}(T)$  be the result of a jeu-de-taquin slide on  $T$  starting from  $C$ , and set  $\text{jdt}(0) := 0$ . Then  $e_i \cdot \text{jdt}(T) = \text{jdt}(e_i \cdot T)$  and  $f_i \cdot \text{jdt}(T) = \text{jdt}(f_i \cdot T)$  for all  $T \in B_{\kappa/\iota}$  and  $1 \leq i \leq s-1$ .

**Corollary 3.16.** *Let  $\kappa$  and  $\iota$  be partitions such that  $\iota_i \leq \kappa_i$  for all positive parts  $\iota_i$  of  $\iota$ , and let  $T \in B_{\kappa/\iota}$ . Then  $\epsilon_i(T) = \epsilon_i(\text{Rect}(T))$  and  $\phi_i(T) = \phi_i(\text{Rect}(T))$  for all  $1 \leq i \leq s-1$ .*

With *jeu de taquin* at his disposal, Schützenberger [22] introduced an action on standard Young tableaux called promotion. Later generalized to the setting of semistandard tableaux by Shimozono [23], promotion for the purposes of this article is a cyclic action that can be thought of as first turning the 1's in a tableau into 2's, the 2's into 3's, etc., and the  $s$ 's into 1's, followed by rearranging the entries via *jeu de taquin* so that the result remains a valid tableau. Because promotion is derived from *jeu de taquin*, it should be no surprise that it inherits compatibility with the raising and lowering crystal operators.

**Definition 3.17.** Let  $\kappa$  be a partition with  $s$  parts, and let  $T$  be a semistandard tableau of shape  $\kappa$ . The result of *jeu-de-taquin demotion* on  $T$  is determined via the following steps.

- (i) The boxes of  $T$  that contain 1 as an entry are emptied, leaving a skew tableau  $T_1$  with entries in  $\{2, 3, \dots, s\}$ .
- (ii) All the entries of  $T_1$  are decremented by 1, leaving a skew tableau  $T'_1$  with entries in  $\{1, 2, \dots, s-1\}$ .
- (iii)  $T'_1$  is rectified under *jeu de taquin*, leaving a tableau of shape  $\kappa_1$ .
- (iv) A tableau  $U$  of shape  $\kappa$  is obtained from  $\text{Rect}(T'_1)$  by filling each box of the skew diagram  $\kappa/\kappa_1$  with  $s$  as an entry.

*Remark 3.18.* The claim that  $U$  is a tableau is justified because the skew diagram  $\kappa/\kappa_1$  is necessarily a *horizontal strip*. Hence *jeu-de-taquin demotion* is well-defined. For more details, see Shimozono [23].

It should be clear that *jeu-de-taquin demotion* acts on  $B_\kappa$ . In particular, demotion is invertible. To see this, note that the algorithm for a *jeu-de-taquin* slide is reversible if the missing box is predetermined (i.e., we can recover  $T$  from  $\text{jdt}(T)$  if we are told the shape of  $T$  in advance).

**Definition 3.19.** Let  $\kappa$  be a partition with  $s$  parts. *Jeu-de-taquin promotion* is the action on  $B_\kappa$  given by  $U \mapsto T$  for all  $U \in B_\kappa$ , where  $T$  denotes the unique tableau of shape  $\kappa$  with entries in  $\{1, 2, \dots, s\}$  for which the result of *jeu-de-taquin demotion* on  $T$  is  $U$ .

The following proposition encapsulates the relationship between promotion and the crystal structure on tableaux.

**Proposition 3.20** (Shimozono [23]). *Let  $\kappa$  be a partition with  $s$  parts, and let  $\text{pr}: B_\kappa \rightarrow B_\kappa$  be jeu-de-taquin promotion. Then, for all  $T \in B_\kappa$ :*

- (i)  $\text{wt}(\text{pr}(T)) = c_s \cdot \text{wt}(T)$ ;



(ii)  $\text{pr}(e_i \cdot T) = e_{i+1} \cdot \text{pr}(T)$  and  $\text{pr}(f_i \cdot T) = f_{i+1} \cdot \text{pr}(T)$  for all  $1 \leq i \leq s-1$ .

*Remark 3.21.* To interpret property (ii), we tacitly stipulate  $\text{pr}(0) := 0$ .

**Definition 3.22.** Let  $\kappa$  be a partition with  $s$  parts. An action  $\gamma: B_\kappa \rightarrow B_\kappa$  is a *weak promotion operator* on the  $\mathfrak{sl}_s$ -crystal  $B_\kappa$  if it satisfies the properties of *jeu-de-taquin* promotion delineated in Proposition 3.20.

**Theorem 3.23** (Bandlow–Schilling–Thiéry [1]). *Let  $\kappa$  be a partition with  $s$  parts, and let  $\text{pr}: B_\kappa \rightarrow B_\kappa$  be jeu-de-taquin promotion. Then  $\text{pr}$  is the unique weak promotion operator on the  $\mathfrak{sl}_s$ -crystal  $B_\kappa$ .*

The following theorem reveals why we restrict our attention to rectangular partitions in the statement of Theorem 1.2.

**Definition 3.24.** A partition  $\kappa$  is *rectangular* if all its positive parts are equal.

**Theorem 3.25** (Shimozono [23]). *Let  $\kappa$  be a partition with  $s$  parts, and let  $\text{pr}: B_\kappa \rightarrow B_\kappa$  be jeu-de-taquin promotion. Then  $\text{pr}^s$  acts as the identity if and only if  $\kappa$  is rectangular.*

*Remark 3.26.* Together, Theorems 3.23 and 3.25 testify at once to the potency of our techniques for investigating rectangular tableaux and to the inherent difficulty in extending them beyond the rectangular setting. Indeed, to address the general case in accordance with the cyclic sieving paradigm, we require a cyclic action of order  $s$  on  $B_\kappa$ . Theorem 3.23 tells us that the only cyclic action compatible with the crystal operators (at least in the way we understand compatibility) is *jeu-de-taquin* promotion, but, by Theorem 3.25, promotion is of the correct order if and only if  $\kappa$  is rectangular.

We proceed to define the Schützenberger involution, which also originates in Schützenberger [22], and may be referred to as evacuation. While promotion and demotion cycle the content of a tableau, evacuation reverses the content: It may be thought of as turning the 1’s in a tableau into  $s$ ’s, the 2’s into  $s-1$ ’s, etc., via a concatenation of  $s-1$  demotions, corresponding to the canonical decomposition of the long element in  $\mathfrak{S}_s$  into a product of  $s-1$  cycles with one descent each, viz.,  $w_0 = (21)(321) \cdots (s \cdots 21)$ . Being derived from *jeu-de-taquin* demotion, the Schützenberger involution, too, inherits compatibility with the raising and lowering crystal operators.

**Definition 3.27.** Let  $\kappa$  be a partition with  $s$  parts, and let  $T$  be a semi-standard tableau of shape  $\kappa$ . The image of  $T$  under the *Schützenberger involution* is determined in the following steps.

- (i)  $s$  is designated the “entry upper bound”;  $T$  is designated the “tableau under examination,” and  $\kappa$  is designated the “shape under consideration.”
- (ii) The boxes in the tableau under examination with entries that exceed the entry upper bound are removed from the tableau.

- (iii) Viewed as a filling of a Young diagram with number of parts equal to the entry upper bound, the resulting tableau is subjected to *jeu-de-taquin* demotion.
- (iv) The product of demotion is designated the “tableau under examination.”
- (v) The shape of the tableau under examination is designated the “shape under consideration.”
- (vi) The shape under consideration is notated, and denoted by  $\kappa_i$ , where  $i$  denotes the entry upper bound.
- (vii) The “entry upper bound” is decremented by 1.
- (viii) The process in steps (ii)-(vi) is iterated until the entry upper bound reaches 0, at which point  $\kappa_0$  is set to be the empty partition, and a tableau is obtained from the increasing sequence  $\emptyset = \kappa_0 \subset \kappa_1 \subset \kappa_2 \subset \cdots \subset \kappa_s = \kappa$  by filling each box of the skew diagram  $\kappa_i/\kappa_{i-1}$  with  $i$  as an entry, for all  $1 \leq i \leq s$ .

**Definition 3.28.** Let  $\kappa$  be a partition with  $s$  parts, and let  $T$  be a semistandard tableau of shape  $\kappa$ . We say that  $T$  is *self-evacuating* if it is stable under the Schützenberger involution.

*Remark 3.29.* It is well-known that evacuation, as defined in Definition 3.27, is an involution (see, for instance, Fulton [7], Appendix A). It is, however, customary to choose the Schensted insertion algorithm as the starting point for the definition of evacuation, rather than the Schützenberger *jeu-de-taquin* slide algorithm, in part because the involutive nature of the insertion-based procedure is easier to see. Such a choice might make more sense if our study of the Schützenberger involution were in isolation, but we insist on *jeu de taquin* because it is the more meaningful of the two combinatorial constructions in the context of crystals.

The following proposition encapsulates the relationship between evacuation and the crystal structure on tableaux.

**Proposition 3.30** (Lascoux–Leclerc–Thibon [13]). *Let  $\kappa$  be a partition with  $s$  parts, and let  $\xi: B_\kappa \rightarrow B_\kappa$  be the Schützenberger involution. Then, for all  $T \in B_\kappa$ :*

- (i)  $\text{wt}(\xi(T)) = w_0 \cdot \text{wt}(T)$ ;
- (ii)  $\xi(e_i \cdot T) = f_{s-i} \cdot \xi(T)$  and  $\xi(f_i \cdot T) = e_{s-i} \cdot \xi(T)$  for all  $1 \leq i \leq s-1$ .

*Remark 3.31.* To interpret property (ii), we tacitly stipulate  $\xi(0) := 0$ .

Just as the properties in Proposition 3.20 characterize promotion, the properties in Proposition 3.30 uniquely determine evacuation.

**Theorem 3.32** (Henriques–Kamnitzer [8]). *Let  $\kappa$  be a partition with  $s$  parts, and let  $\xi: B_\kappa \rightarrow B_\kappa$  be the Schützenberger involution. If an action  $\gamma: B_\kappa \rightarrow B_\kappa$  satisfies the properties of evacuation delineated in Proposition 3.30, then  $\gamma$  and  $\xi$  are coincident.*

Thus, evacuation is in some sense the only involution on tableaux compatible with the crystal operators, which explains why it serves us well as a choice of order  $n$  cyclic action in the case  $n = 2$ .

Finally, as promised, we reinterpret the Yamanouchi conditions on reading words as vanishing conditions on crystal operators. Because the following propositions are essentially self-evident, we omit the proofs.

**Proposition 3.33.** *Let  $\kappa$  be a partition with  $s$  parts, and let  $T$  be a tableau of shape  $\kappa$ . For all  $1 \leq i \leq s - 1$ , the word of  $T$  is Yamanouchi (anti-Yamanouchi) with respect to the integers  $i$  and  $i + 1$  if and only if the raising operator  $e_i$  (lowering operator  $f_i$ ) vanishes at  $T$ .*

**Proposition 3.34.** *Let  $\kappa$  be a partition with  $s$  parts, and let  $T$  be a tableau of shape  $\kappa$ . For all  $1 \leq i < i' \leq s - 1$ , the word of  $T$  is Yamanouchi (anti-Yamanouchi) in the subset  $\{i, i + 1, \dots, i'\}$  if and only if the raising operators  $e_i, e_{i+1}, \dots, e_{i'-1}$  (lowering operators  $f_i, f_{i+1}, \dots, f_{i'-1}$ ) all vanish at  $T$ .*

These propositions put us in a position to show that the tableaux sets  $\text{EYTab}(\lambda, \bar{\mu}\mu)$  and  $\text{PYTab}(\lambda, \mu^n)$  are in bijection with those specified by the Littlewood–Richardson rule.

**Definition 3.35.** Let  $\lambda$  be a partition with  $2m$  parts, and let  $\mu$  be a partition of  $|\lambda|/2$  with  $m$  parts such that  $\mu_i \leq \lambda_i$  for all positive parts  $\mu_i$  of  $\mu$ . Denote the composition  $(\mu_m, \mu_{m-1}, \dots, \mu_1, \mu_1, \mu_2, \dots, \mu_m)$  by  $\bar{\mu}\mu$ .  $\text{Tab}(\lambda, \bar{\mu}\mu)$  is the set of semistandard tableaux of shape  $\lambda$  and content  $\bar{\mu}\mu$ , and  $\text{EYTab}(\lambda, \bar{\mu}\mu)$  is the subset of  $\text{Tab}(\lambda, \bar{\mu}\mu)$  consisting of those tableaux with reading word anti-Yamanouchi in  $\{1, 2, \dots, m\}$  and Yamanouchi in  $\{m+1, m+2, \dots, 2m\}$ .

**Proposition 3.36.** *Let  $T$  be a tableau with entries in  $\{1, 2, \dots, m\}$  and anti-Yamanouchi reading word. Then  $T$  is of shape  $\mu$  if and only if  $T$  is of content  $\bar{\mu}$ .*

*Proof.* Let  $\nu$  be the shape of  $T$ , and let  $B_\nu$  be the set of semistandard tableaux of shape  $\nu$  equipped with the structure of an  $\mathfrak{sl}_m$ -crystal in accordance with Proposition 3.9. Recall from Proposition 3.10 that  $B_\nu$  is a highest weight crystal in which the unique highest weight element is of content  $\nu$ . By analogous reasoning,  $B_\nu$  is a lowest weight crystal in which the unique lowest weight element is of content  $\bar{\nu}$ . Since the word of  $T$  is anti-Yamanouchi in  $\{1, 2, \dots, m\}$ , it follows that  $T$  is the unique lowest weight element in  $B_\nu$ , so this completes the proof.  $\square$

**Proposition 3.37.** *For all  $T \in \text{EYTab}(\lambda, \bar{\mu}\mu)$ , let  $T'$  be the skew tableau obtained from  $T$  by removing each box with an entry in  $\{1, 2, \dots, m\}$  and decrementing the entry in each remaining box by  $m$ . Then the association  $T \mapsto T'$  defines a bijection from  $\text{EYTab}(\lambda, \bar{\mu}\mu)$  to the set of semistandard skew tableaux of shape  $\lambda/\mu$  and content  $\mu$  with Yamanouchi reading word.*

*Proof.* Let  $T \in \text{EYTab}(\lambda, \bar{\mu}\mu)$ , and let  $\tilde{T}$  be the tableau obtained from  $T$  by removing each box with an entry not in  $\{1, 2, \dots, m\}$ . Then  $\tilde{T}$  is the unique

tableau of shape  $\mu$  and content  $\bar{\mu}$  (cf. Proposition 3.36). Therefore,  $T'$  is of shape  $\lambda/\mu$  and content  $\mu$ ; its reading word is Yamanouchi, and the claim that the association  $T \mapsto T'$  is bijective follows immediately.  $\square$

**Corollary 3.38.**  $|\text{EYTab}(\lambda, \bar{\mu}\mu)| = \langle s_\mu^2, s_\lambda \rangle$ .

*Proof.* In view of Proposition 3.37, this is the Littlewood–Richardson rule.  $\square$

**Definition 3.39.** Let  $\lambda$  be a partition with  $mn$  parts, and let  $\mu$  be a partition of  $|\lambda|/n$  with  $m$  parts such that  $\mu_i \leq \lambda_i$  for all positive parts  $\mu_i$  of  $\mu$ . Denote the composition  $(\mu_1, \mu_2, \dots, \mu_m, \mu_1, \mu_2, \dots, \mu_m, \dots, \mu_1, \mu_2, \dots, \mu_m)$  by  $\mu^n$ .  $\text{Tab}(\lambda, \mu^n)$  is the set of semistandard tableaux of shape  $\lambda$  and content  $\mu^n$ , and  $\text{PYTab}(\lambda, \mu^n)$  is the subset of  $\text{Tab}(\lambda, \mu^n)$  consisting of those tableaux with reading word Yamanouchi in  $\{km + 1, km + 2, \dots, (k + 1)m\}$  for all  $0 \leq k \leq n - 1$ .

**Proposition 3.40.** *Let  $T$  be a tableau with entries in  $\{1, 2, \dots, m\}$  and Yamanouchi reading word. Then  $T$  is of shape  $\mu$  if and only if  $T$  is of content  $\mu$ .*

*Proof.* Let  $\nu$  be the shape of  $T$ , and let  $B_\nu$  be the set of semistandard tableaux of shape  $\nu$  equipped with the structure of an  $\mathfrak{sl}_m$ -crystal in accordance with Proposition 3.9. Recall from Proposition 3.10 that  $B_\nu$  is a highest weight crystal in which the unique highest weight element is of content  $\nu$ . Since the word of  $T$  is Yamanouchi in  $\{1, 2, \dots, m\}$ , it follows that  $T$  is the unique highest weight element of  $B_\nu$ , so this completes the proof.  $\square$

**Proposition 3.41.** *For all  $T \in \text{PYTab}(\lambda, \mu^n)$ , let  $T'$  be the skew tableau obtained from  $T$  by removing each box with an entry in  $\{1, 2, \dots, m\}$  and decrementing the entry in each remaining box by  $m$ . Then the association  $T \mapsto T'$  defines a bijection from  $\text{PYTab}(\lambda, \mu^n)$  to the set of semistandard skew tableaux of shape  $\lambda/\mu$  and content*

$$\mu^{n-1} := (\mu_1, \mu_2, \dots, \mu_m, \mu_1, \mu_2, \dots, \mu_m, \dots, \mu_1, \mu_2, \dots, \mu_m)$$

*with reading word Yamanouchi in  $\{km + 1, km + 2, \dots, (k + 1)m\}$  for all  $0 \leq k \leq n - 1$ .*

*Proof.* Let  $T \in \text{PYTab}(\lambda, \mu^n)$ , and let  $\tilde{T}$  be the tableau obtained from  $T$  by removing each box with an entry not in  $\{1, 2, \dots, m\}$ . Then  $\tilde{T}$  is the unique tableau of shape  $\mu$  and content  $\mu$  (cf. Proposition 3.40). Therefore,  $T'$  is of shape  $\lambda/\mu$  and content  $\mu^{n-1}$ ; its reading word is Yamanouchi in  $\{km + 1, km + 2, \dots, (k + 1)m\}$  for all  $0 \leq k \leq n - 1$ , and the claim that the association  $T \mapsto T'$  is bijective follows immediately.  $\square$

**Corollary 3.42.**  $|\text{PYTab}(\lambda, \mu^n)| = \langle s_\mu^n, s_\lambda \rangle$ .

*Proof.* By induction on  $n$ , we see that

$$\langle s_\mu^n, s_\lambda \rangle = \sum_{\theta_2, \theta_3, \dots, \theta_{n-1}} \langle s_\mu^2, s_{\theta_2} \rangle \langle s_{\theta_2} s_\mu, s_{\theta_3} \rangle \cdots \langle s_{\theta_{n-1}} s_\mu, s_\lambda \rangle,$$

where the sum ranges over all increasing sequences of partitions

$$\mu \subset \theta_2 \subset \theta_3 \subset \cdots \subset \theta_{n-1} \subset \lambda.$$

In view of Proposition 3.41, the identification of  $|\text{PYTab}(\lambda, \mu^n)|$  with the right-hand side is the Littlewood–Richardson rule.  $\square$

#### 4. PROOFS OF THEOREMS 1.1 AND 1.2

In this section, we prove our main theorems. We start with an overview of the Kazhdan–Lusztig basis for the Hecke algebra  $H_t(q)$ , and we describe a collection of corresponding bases introduced by Skandera [24] for the irreducible polynomial representations of  $GL_s(C)$ . For  $\kappa$  a partition with at most  $s$  positive parts, we consider  $\kappa$  as a partition with  $s$  parts, and we reprise the argument of Rhoades [20] that the action of the long cycle  $c_s \in \mathfrak{S}_s \subset GL_s(C)$  on the Kazhdan–Lusztig immanants generating a  $GL_s(\mathbb{C})$ -representation associated to  $\kappa$  lifts (up to sign) *jeu-de-taquin* promotion on the tableaux in the  $\mathfrak{sl}_s$ -crystal  $B_\kappa$  if  $\kappa$  is rectangular, and we present the analogous argument that the action of the long element  $w_0 \in \mathfrak{S}_s \subset GL_s(C)$  on immanants lifts (up to sign) the Schützenberger involution for all  $\kappa$ . Setting  $s := mn$ , we derive the desired conclusions from character computations, drawing extensively upon the background developed in the two preceding sections. Some familiarity with the combinatorics of  $\mathfrak{S}_t$  and the character theory of  $GL_s(C)$  is assumed.<sup>3</sup>

**4.1. The Basis of Kazhdan–Lusztig Immanants.** The Hecke algebra  $H_t(q)$  is a quantum deformation of the symmetric group algebra  $\mathbb{C}[\mathfrak{S}_t]$  that reduces to  $\mathbb{C}[\mathfrak{S}_t]$  in the limit  $q \rightarrow 1$ . Skandera’s construction begins with the Kazhdan–Lusztig polynomials  $P_{v,w}(q) \in H_t(q)$  associated to (ordered) pairs of permutations  $v, w \in \mathfrak{S}_t$ .

**Definition 4.1.** Let  $\mathbb{C}[\{x_{i,j}\}]$  be the ring of polynomials with complex coefficients in the  $t^2$  variables  $\{x_{i,j}\}_{1 \leq i,j \leq t}$ . A polynomial in  $\mathbb{C}[\{x_{i,j}\}]$  is an *immanant* if it can be expressed as a linear combination of the *permutation monomials*  $\{x_{1,w(1)}x_{2,w(2)} \cdots x_{t,w(t)}\}_{w \in \mathfrak{S}_t}$ .

**Definition 4.2.** Let the variable set  $\{x_{i,j}\}_{1 \leq i,j \leq t}$  be denoted by  $x$ . For all  $w \in \mathfrak{S}_t$ , the *Kazhdan–Lusztig immanant* associated to  $w$  is the polynomial

$$\text{Imm}_w(x) := \sum_{v \geq w} (-1)^{\ell(v) - \ell(w)} P_{w_0 v, w_0 w}(1) x_{1,v(1)} x_{2,v(2)} \cdots x_{t,v(t)},$$

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<sup>3</sup>The discussion of the algebraic tools introduced in this section is necessarily abbreviated. For more details on the irreducible polynomial characters of  $GL_s(C)$ , consult Fulton [7], Chapter 8. For more about the Kazhdan–Lusztig basis, see the original paper by Kazhdan and Lusztig [12], or see Björner–Brenti [4] for an expository account that addresses the combinatorics of  $\mathfrak{S}_t$  as well. The crucial facts concerning the Skandera bases may be found in Rhoades–Skandera [21] and Skandera [24]. The entire section is informed by the recent article “Cyclic sieving, promotion, and representation theory” by Brendon Rhoades [20], to which a considerable intellectual debt is owed and appreciated.

where the sum is taken over all  $v \in \mathfrak{S}_t$  such that  $v \geq w$  in the Bruhat order.

Note that a composition  $\eta$  of  $t$  with  $s$  parts determines a weakly increasing function  $\{1, 2, \dots, t\} \rightarrow \{1, 2, \dots, s\}$ , which we also denote by  $\eta$ , given by  $\eta(j) = i$  for all  $\eta_1 + \eta_2 + \dots + \eta_{i-1} + 1 \leq j \leq \eta_1 + \eta_2 + \dots + \eta_i$ . The association  $x_{i,j} \mapsto x_{\eta(i),j}$  defines a  $\mathbb{C}$ -algebra homomorphism  $\pi_\eta: \mathbb{C}[\{x_{i,j}\}] \rightarrow \mathbb{C}[\{x_{\eta(i),j}\}]$ . Identifying the polynomial ring  $\mathbb{C}[\{x_{\eta(i),j}\}]$  with the symmetric algebra  $\text{Sym}((\mathbb{C}^s)^* \otimes (\mathbb{C}^t)^*)$ , we may construe  $GL_s(\mathbb{C})$  to act on  $\mathbb{C}[\{x_{\eta(i),j}\}]$  via its canonical action on  $(\mathbb{C}^s)^*$ .

For all  $w \in \mathfrak{S}_t$ , we denote the image of the immanant  $\text{Imm}_w(x)$  by  $\text{Imm}_w(x_\eta)$ . Recalling that the Robinson–Schensted–Knuth correspondence (cf. Fulton [7], Chapter 4) gives a bijection between permutations in  $\mathfrak{S}_t$  and (ordered) pairs of standard tableaux with entries  $1, 2, \dots, t$ , we assign immanants to pairs of standard tableaux in place of permutations. To build representations of  $GL_s(\mathbb{C})$  out of immanants indexed by semistandard tableaux, we appeal to processes for interchanging standard and semistandard tableaux, which we refer to as standardization and semistandardization.

*Remark 4.3.* To preserve compatibility with Rhoades [20], we consider semistandard tableaux to be row-strict rather than column-strict, i.e. a tableau is semistandard if its entries are strictly increasing across rows and weakly increasing down columns. We make this deviation from accepted terminology in this subsection only.

**Definition 4.4.** Let  $\eta$  be a composition of  $t$  with  $s$  parts, and let  $T$  be a semistandard tableau of content  $\eta$ . The *standardization* of  $T$ , which we denote by  $\text{std}(T)$ , is the tableau obtained from  $T$  by replacing the  $\eta_i$  instances of  $i$  in  $T$  with one instance of each  $j$  for which  $\eta(j) = i$ , for all  $1 \leq i \leq s$ , in such a way that if  $1 \leq j < j' \leq t$  and  $\eta(j) = \eta(j')$ , then the row in which  $j$  occurs is higher than the row in which  $j'$  occurs.

**Definition 4.5.** Let  $\eta$  be a composition of  $t$  with  $s$  parts, and let  $T$  be a standard tableau with entries  $1, 2, \dots, t$ . We say that  $T$  is  $\eta$ -semistandardizable if, for all  $1 \leq j < j' \leq t$  for which  $\eta(j) = \eta(j')$ , the row in which  $j$  occurs is higher than the row in which  $j'$  occurs. In this case, the  $\eta$ -semistandardization of  $T$ , which we denote by  $\text{rst}_\eta(T)$ , is the tableau obtained from  $T$  by replacing each instance of  $j$  with an instance of  $\eta(j)$  for all  $1 \leq j \leq t$ .

Note that standardization gives a bijection between the semistandard tableaux of shape  $\kappa$  and content  $\eta$  and the  $\eta$ -semistandardizable standard tableaux of shape  $\kappa$ , the inverse of which is given by  $\eta$ -semistandardization.

At this point, we are ready to discuss Skandera's construction (which had its origins in work of Du [6]). Let  $\kappa$  be a partition of  $t$ , and let  $T$  be a standard tableau of shape  $\kappa$ . We denote by  $V_{T,s}$  a particular quotient space (defined in Rhoades [20]) of the complex vector space generated by the immanants  $\text{Imm}_w(x_\eta)$ , where  $w$  ranges over  $\mathfrak{S}_t$ , and  $\eta$  ranges over all

compositions of  $t$  with  $s$  parts. Lemma 4.5 of Rhoades [20] tells us that  $V_{T,s}$  inherits the structure of a  $GL_s(\mathbb{C})$ -representation from the  $GL_s(\mathbb{C})$ -action on  $\mathbb{C}[\{x_{\eta(i),j}\}]$ , and the essential properties of  $V_{T,s}$  are encapsulated in the following theorem.

**Theorem 4.6** (Rhoades [20]). *For all compositions  $\eta$  of  $t$  with  $s$  parts and semistandard tableaux  $U'$  of shape  $\kappa$  and content  $\eta$ , let  $I_\eta(U')$  be the image of  $\text{Imm}_{(T, \text{std}(U'))}(x_\eta)$  in  $V_{T,s}$ . Set*

$$I_\eta := \{I_\eta(U') : U' \text{ is a semistandard tableau of shape } \kappa \text{ and content } \eta\}.$$

*Then the following three claims hold.*

- (i) *As a  $GL_s(\mathbb{C})$ -representation,  $V_{T,s}$  is isomorphic to the dual of the irreducible polynomial  $GL_s(\mathbb{C})$ -representation with highest weight  $\kappa'$ , where  $\kappa'$  denotes the conjugate partition to  $\kappa$ .*
- (ii) *The set  $\bigcup_\eta I_\eta$ , where  $\eta$  ranges over all compositions of  $t$  with  $s$  parts, constitutes a basis for  $V_{T,s}$ .*
- (iii) *For all compositions  $\eta$  of  $t$  with  $s$  parts, the set  $I_\eta$  constitutes a basis for the weight space of  $V_{T,s}$  corresponding to the weight  $-\eta$ , which we denote by  $V_{T,s,\eta}$ .*

We proceed to verify that the actions of  $w_0$  and  $c_s$  on the Skandera basis of  $V_{T,s}$  indeed lift the actions of evacuation and promotion, respectively, on the semistandard tableaux of shape  $\kappa$ . We start with the case of standard tableaux.

**Definition 4.7.** Let  $\iota$  be a partition with  $a$  positive parts. We write  $v(\iota)$  for the sum  $\sum_{i=1}^a (i-1)\iota_i$ .

**Lemma 4.8.** *Let  $\kappa$  be a partition of  $t$ , and let  $P$  be a standard tableau of shape  $\kappa$ . Let  $w_0$  be the long element in  $\mathfrak{S}_t$ , and let  $\xi$  be the Schützenberger involution. Then*

$$w_0 \cdot I_{1^t}(P) = (-1)^{v(\kappa')} \cdot I_{1^t}(\xi(P)).$$

*Proof.* We borrow the reasoning from the proof of Lemma 5.2 in Rhoades [20]. It suffices to show the analogous result holds for the action of  $w_0$  on the Kazhdan–Lusztig basis of the left cellular representation of  $\mathfrak{S}_t$  associated to  $\kappa$ . Theorem 5.1 of Stembridge [27] tells us that  $w_0$  lifts the action of  $\xi$  up to (global) sign. From Theorem 4.3 of Stembridge [27], it follows that the sign must be  $(-1)^{v(\kappa')}$ , as desired.  $\square$

*Remark 4.9.* It might seem like the sign should be  $(-1)^{v(\kappa)}$ , but to extrapolate from the cellular Kazhdan–Lusztig basis to the basis of Kazhdan–Lusztig immanants, it is necessary to conjugate the indexing partitions.

**Lemma 4.10.** *Let  $\kappa$  be a rectangular partition of  $t$  with  $\kappa_1 = b$ , and let  $P$  be a standard tableau of shape  $\kappa$ . Let  $c_t$  be the long cycle in  $\mathfrak{S}_t$ , and let  $\text{pr}$  be jeu-de-taquin promotion. Then*

$$c_t \cdot I_{1^t}(P) = (-1)^{b-1} \cdot I_{1^t}(\text{pr}(P)).$$

*Proof.* This is Lemma 5.2 of Rhoades [20].  $\square$

To generalize Lemmas 4.8 and 4.10 to the semistandard setting, we first investigate the interactions between the Schützenberger actions and semistandardization.

**Lemma 4.11.** *Let  $\kappa$  be a partition of  $t$ , and let  $P$  be a standard tableau of shape  $\kappa$ . Let  $\xi$  be the Schützenberger involution. For all compositions  $\eta$  of  $t$  with  $s$  parts,  $P$  is  $\eta$ -semistandardizable if and only if  $\xi(P)$  is  $(w_0 \cdot \eta)$ -semistandardizable. If  $P$  is  $\eta$ -semistandardizable, then*

$$\xi(\text{rst}_\eta(P)) = \text{rst}_{w_0 \cdot \eta}(\xi(P)).$$

*Proof.* From the definition of evacuation via Schensted insertion (cf. Stembridge [27]), it should be clear that it suffices to show the first claim. Note that  $P$  is  $\eta$ -semistandardizable if and only if  $\eta_1 + \eta_2 + \cdots + \eta_{i-1} < j < \eta_1 + \eta_2 + \cdots + \eta_i$  implies that  $j$  belongs to the descent set of  $P$ , for all  $1 \leq i \leq s$ . Since  $j$  belongs to the descent set of  $P$  if and only if  $t - j$  belongs to the descent set of  $\xi(P)$  (cf. Theorem 6.6.2 of Björner–Brenti [4]), the desired conclusion follows.  $\square$

**Lemma 4.12.** *Let  $\kappa$  be a partition of  $t$ , and let  $P$  be a standard tableau of shape  $\kappa$ . Let  $\text{pr}$  be jeu-de-taquin promotion. For all compositions  $\eta$  of  $t$  with  $s$  parts,  $P$  is  $\eta$ -semistandardizable if and only if  $\text{pr}^{\eta_s}(P)$  is  $(c_s \cdot \eta)$ -semistandardizable. If  $P$  is  $\eta$ -semistandardizable, then*

$$\text{pr}(\text{rst}_\eta(P)) = \text{rst}_{c_s \cdot \eta}(\text{pr}^{\eta_s}(P)).$$

*Proof.* This is Lemma 5.3 of Rhoades [20].  $\square$

Next, we identify the corresponding relationships between the long element and long cycle actions and the projections that convert “standard” immanants into “semistandard” ones. In particular, for all compositions  $\eta$  of  $t$  with  $s$  parts, the following diagrams are commutative.

$$\begin{array}{ccc} \mathbb{C}[\{x_{i,j}\}] & \xrightarrow{w_0 \in \mathfrak{S}_t} & \mathbb{C}[\{x_{i,j}\}] & \mathbb{C}[\{x_{i,j}\}] & \xrightarrow{c_t^{\eta_s} \in \mathfrak{S}_t} & \mathbb{C}[\{x_{i,j}\}] \\ \pi_\eta \downarrow & & \downarrow \pi_{w_0 \cdot \eta} & \pi_\eta \downarrow & & \downarrow \pi_{c_s \cdot \eta} \\ \mathbb{C}[\{x_{\eta(i),j}\}] & \xrightarrow{w_0 \in \mathfrak{S}_s} & \mathbb{C}[\{x_{(w_0 \cdot \eta)(i),j}\}] & \mathbb{C}[\{x_{\eta(i),j}\}] & \xrightarrow{c_s \in \mathfrak{S}_s} & \mathbb{C}[\{x_{(c_s \cdot \eta)(i),j}\}] \end{array}$$

In view of Theorem 4.6, we may conclude that the following diagrams are also commutative.

$$\begin{array}{ccc} V_{T,t,1^t} & \xrightarrow{w_0 \in \mathfrak{S}_t} & V_{T,t,1^t} & V_{T,t,1^t} & \xrightarrow{c_t^{\eta_s} \in \mathfrak{S}_t} & V_{T,t,1^t} \\ \pi_\eta \downarrow & & \downarrow \pi_{w_0 \cdot \eta} & \pi_\eta \downarrow & & \downarrow \pi_{c_s \cdot \eta} \\ V_{T,s,\eta} & \xrightarrow{w_0 \in \mathfrak{S}_s} & V_{T,s,w_0 \cdot \eta} & V_{T,s,\eta} & \xrightarrow{c_s \in \mathfrak{S}_s} & V_{T,s,c_s \cdot \eta} \end{array}$$

Finally, we come to the following theorems.



**Theorem 4.13.** *Let  $\kappa$  be a partition of  $t$ , and let  $\eta$  be a composition of  $t$  with  $s$  parts. Let  $U'$  be a semistandard tableau of shape  $\kappa$  and content  $\eta$ . Let  $w_0$  be the long element in  $\mathfrak{S}_s$ , and let  $\xi$  be the Schützenberger involution. Then*

$$w_0 \cdot I_\eta(U') = (-1)^{v(\kappa')} \cdot I_{w_0 \cdot \eta}(\xi(U')).$$

*Proof.* The following computation is carried out with reference to the above commutative diagrams as well as to Lemmas 4.8 and 4.11.

$$\begin{aligned} w_0 \cdot I_\eta(U') &= \pi_{w_0 \cdot \eta} \cdot w_0 \cdot I_{1^t}(\text{std}(U')) \\ &= \pi_{w_0 \cdot \eta} \cdot (-1)^{v(\kappa')} \cdot I_{1^t}(\xi(\text{std}(U'))) \\ &= (-1)^{v(\kappa')} \cdot I_{w_0 \cdot \eta}(\text{rst}_{w_0 \cdot \eta}(\xi(\text{std}(U')))) \\ &= (-1)^{v(\kappa')} \cdot I_{w_0 \cdot \eta}(\xi(\text{rst}_\eta(\text{std}(U')))) \\ &= (-1)^{v(\kappa')} \cdot I_{w_0 \cdot \eta}(\xi(U')). \end{aligned}$$

□

**Theorem 4.14.** *Let  $\kappa$  be a rectangular partition of  $t$  with  $\kappa_1 = b$ , and let  $\eta$  be a composition of  $t$  with  $s$  parts. Let  $U'$  be a semistandard tableau of shape  $\kappa$  and content  $\eta$ . Let  $c_s$  be the long cycle in  $\mathfrak{S}_s$ , and let  $\text{pr}$  be jeu-de-taquin promotion. Then*

$$c_s \cdot I_\eta(U') = (-1)^{\eta_s(b-1)} I_{c_s \cdot \eta}(\text{pr}(U')).$$

*Proof.* This is Proposition 5.5 of Rhoades [20], and the proof is analogous to our proof of Theorem 4.13. □

**4.2. Proof of Theorem 1.1.** Let  $\lambda$  be a partition with  $2m$  parts. Suppose that 2 divides  $|\lambda|$ , and let  $\mu = (\mu_1, \mu_2, \dots, \mu_m)$  be a partition of  $|\lambda|/2$ . Let  $B_\lambda$  be the set of semistandard tableaux of shape  $\lambda$ , endowed with an  $\mathfrak{sl}_{2m}$ -crystal structure by the sets of raising and lowering operators  $\{e_i\}_{i=1}^{2m-1}$  and  $\{f_i\}_{i=1}^{2m-1}$ , respectively. The key to our proof is the assignment of an  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal structure to  $B_\lambda$  that allows us to inspect the action of  $\xi$  on its connected components. This process provides a combinatorial model for the decomposition into irreducible components of the restriction to  $GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$  of the irreducible  $GL_{2m}(\mathbb{C})$ -representation with highest weight  $\lambda$ , which underlies our character evaluation.

Recall that we chose  $\{E_1 - E_2, E_2 - E_3, \dots, E_{2m-1} - E_{2m}\}$  as the set of simple roots for  $\mathfrak{sl}_{2m}$ . Here we choose  $\{E_2 - E_1, E_3 - E_2, \dots, E_m - E_{m-1}, E_{m+1} - E_{m+2}, E_{m+2} - E_{m+3}, \dots, E_{2m-1} - E_{2m}\}$  as the set of simple roots for  $\mathfrak{sl}_m \oplus \mathfrak{sl}_m$ .

**Proposition 4.15.** *The set  $B_\lambda$  equipped with the map  $\text{wt}$ , the set of raising operators  $\{f_1, f_2, \dots, f_{m-1}, e_{m+1}, e_{m+2}, \dots, e_{2m-1}\}$ , and the set of lowering operators  $\{e_1, e_2, \dots, e_{m-1}, f_{m+1}, f_{m+2}, \dots, f_{2m-1}\}$  is a regular  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal.*

*Proof.* It is a simple matter to verify that the conditions of Definition 3.1 hold for  $\mathfrak{g} = \mathfrak{sl}_m \oplus \mathfrak{sl}_m$  with the indicated choice of simple roots. Hence  $B_\lambda$

is an  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal. To establish regularity, note that drawing any two operators from distinct sets among  $\{e_1, e_2, \dots, e_{m-1}, f_1, f_2, \dots, f_{m-1}\}$  and  $\{e_{m+1}, e_{m+2}, \dots, e_{2m-1}, f_{m+1}, f_{m+2}, \dots, f_{2m-1}\}$  yields a commuting pair, so it suffices to show conditions (iv) - (vib) of Definition 3.3 hold for all  $i \neq j$  such that  $1 \leq i, j \leq m-1$  or  $m+1 \leq i, j \leq 2m-1$ . In the latter case, this is immediate because  $B_\lambda$  is regular as an  $\mathfrak{sl}_{2m}$ -crystal, and, in the former case, it follows in view of the additional observation that interchanging  $e_i$  and  $f_i$  for all  $1 \leq i \leq m-1$  interchanges  $\Delta_i \epsilon_j$  with  $\nabla_i \phi_j$  and  $\Delta_i \phi_j$  with  $\nabla_i \epsilon_j$  for all  $1 \leq i, j \leq m-1$  (cf. Stembridge [28], p. 4809).  $\square$

Each tableau in  $B_\lambda$  is made up of two “subtableaux”: a tableau with entries in  $\{1, 2, \dots, m\}$  and a skew tableau with entries in  $\{m+1, m+2, \dots, 2m\}$ . These subtableaux do not interact with each other under any of the raising and lowering  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal operators, so it is worthwhile to consider them independently.

**Definition 4.16.** For all tableaux  $T \in B_\lambda$ , let  $\varphi_0(T)$  be the tableau obtained from  $T$  by removing each box with an entry not in  $\{1, 2, \dots, m\}$ , and let  $\varphi_1(T)$  be the skew tableau obtained from  $T$  by removing each box with an entry not in  $\{m+1, m+2, \dots, 2m\}$ , and reducing modulo  $m$  the entry in each remaining box, so that the entries of  $\varphi_1(T)$  are also among  $1, 2, \dots, m$ . Let  $\varphi(T)$  be the ordered pair of tableaux  $(\varphi_0(T), \text{Rect}(\varphi_1(T)))$ .

**Proposition 4.17.** *Let  $\mathcal{C}$  be a connected component of the  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal  $B_\lambda$ . Then  $\mathcal{C}$  is a highest weight crystal. Furthermore, if  $b$  is the unique highest weight element of  $\mathcal{C}$ , then there exist partitions  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$  and  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$  such that  $b$  is of content  $\bar{\beta}\gamma$  and  $\varphi(b) = (b_{\bar{\beta}}, b_\gamma)$ , where  $b_{\bar{\beta}}$  is the unique tableau of shape  $\beta$  and content  $\bar{\beta}$ , and  $b_\gamma$  is the unique tableau of shape  $\gamma$  and content  $\gamma$ .*

*Proof.* Since  $\mathcal{C}$  is a regular connected crystal, it follows from Proposition 3.7 that  $\mathcal{C}$  is a highest weight crystal. Let  $b$  be the unique highest weight element of  $\mathcal{C}$ . Recall that  $b$  is anti-Yamanouchi in  $\{1, 2, \dots, m\}$  and Yamanouchi in  $\{m+1, m+2, \dots, 2m\}$ . From Proposition 3.36, we see that there exists a partition  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$  such that  $\varphi_0(b) = b_{\bar{\beta}}$ , and, from Proposition 3.40 (in view of Corollary 3.16), we see that there exists a partition  $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$  such that  $\text{Rect}(\varphi_1(T)) = b_\gamma$ .  $\square$

**Proposition 4.18.** *Let  $\beta$  and  $\gamma$  be partitions, each with  $m$  parts. Equip the set  $B_{(\bar{\beta}, \gamma)} := B_\beta \times B_\gamma$  with the map  $\text{wt} \times \text{wt}$ . For all  $1 \leq i \leq m-1$ , let  $e_i$  and  $f_i$  act as the  $\mathfrak{sl}_m$ -crystal operators  $e_i$  and  $f_i$ , respectively, on  $B_\beta$  and as the identity on  $B_\gamma$ . For all  $m+1 \leq i \leq 2m-1$ , let  $e_i$  and  $f_i$  act as the identity on  $B_\beta$  and as the  $\mathfrak{sl}_m$ -crystal operators  $e_{i-m}$  and  $f_{i-m}$ , respectively, on  $B_\gamma$ . Then  $B_{(\bar{\beta}, \gamma)}$ , together with the set of raising operators  $\{f_1, f_2, \dots, f_{m-1}, e_{m+1}, e_{m+2}, \dots, e_{2m-1}\}$  and the set of lowering operators  $\{e_1, e_2, \dots, e_{m-1}, f_{m+1}, f_{m+2}, \dots, f_{2m-1}\}$ , is a regular connected  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal with unique highest weight element  $(b_{\bar{\beta}}, b_\gamma)$ .*

*Proof.* It is apparent that  $B_{(\bar{\beta}, \gamma)}$  is an  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal. By the reasoning in the proof of Proposition 4.15, to establish regularity it suffices to show conditions (iv) - (vib) of Definition 3.3 hold for all  $i \neq j$  such that  $1 \leq i, j \leq m-1$  or  $m+1 \leq i, j \leq 2m-1$ . In the latter case, this is immediate because  $B_\gamma$  is regular as an  $\mathfrak{sl}_m$ -crystal, and, in the former case, it follows because  $B_\beta$  is regular as an  $\mathfrak{sl}_m$ -crystal in view of the additional observation that interchanging  $e_i$  and  $f_i$  for all  $1 \leq i \leq m-1$  interchanges  $\Delta_i \epsilon_j$  with  $\nabla_i \phi_j$  and  $\Delta_i \phi_j$  with  $\nabla_i \epsilon_j$  for all  $1 \leq i, j \leq m-1$ .

The claim that  $(b_{\bar{\beta}}, b_\gamma)$  is the unique highest weight element of  $B_{(\bar{\beta}, \gamma)}$  is a direct consequence of Propositions 3.36 and 3.40.  $\square$

Thus, if  $\mathcal{C}$  is a connected component of  $B_\lambda$ , there exist partitions  $\beta$  and  $\gamma$  for which the unique highest weight element of  $\mathcal{C}$  corresponds to that of  $B_{(\bar{\beta}, \gamma)}$ . In fact, the two crystals are structurally identical.

**Theorem 4.19.** *Let  $\mathcal{C}$  be a connected component of the  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal  $B_\lambda$ . Let  $b$  be the unique highest weight element of  $\mathcal{C}$ , and let  $\beta$  and  $\gamma$  be partitions, each with  $m$  parts, for which  $\varphi(b) = (b_{\bar{\beta}}, b_\gamma)$ . Then  $\varphi$  restricts to an isomorphism of crystals  $\mathcal{C} \xrightarrow{\sim} B_{(\bar{\beta}, \gamma)}$ .*

*Proof.* Since *jeu-de-taquin* slides commute with crystal operators (cf. Proposition 3.15), it should be clear that  $\varphi|_{\mathcal{C}}: \mathcal{C} \rightarrow B_{(\bar{\beta}, \gamma)}$  is a morphism of crystals. Furthermore, the equality  $\phi_i(b) = \phi_i(b_{\bar{\beta}}, b_\gamma)$  holds for all  $1 \leq i \leq m-1$  by definition of  $\varphi_0$ , and it holds for all  $m+1 \leq i \leq 2m-1$  by definition of  $\varphi_1$  in view of Corollary 3.16. Thus, Proposition 3.8 tells us that  $\mathcal{C}$  and  $B_{(\bar{\beta}, \gamma)}$  are isomorphic. A morphism of crystals  $\mathcal{C} \rightarrow B_{(\bar{\beta}, \gamma)}$  is uniquely determined by its image at  $b$ , so we may conclude that  $\varphi|_{\mathcal{C}}$  is an isomorphism.  $\square$

We turn our attention now to the action of  $\xi$ , first on the highest weight elements of the  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal  $B_\lambda$ , and then on all its tableaux.

**Lemma 4.20.** *Let  $b$  be a highest weight element of  $B_\lambda$ , and let  $\beta$  and  $\gamma$  be partitions, each with  $m$  parts, such that  $\varphi(b) = (b_{\bar{\beta}}, b_\gamma)$ . Then  $\varphi(\xi(b)) = (b_{\bar{\gamma}}, b_\beta)$ .*

*Remark 4.21.* If  $\mu$  is a partition with  $m$  parts, then  $\text{EYTab}(\lambda, \bar{\mu}\mu)$  is the set of highest weight elements of  $B_\lambda$  with content  $\bar{\mu}\mu$ . Thus, Lemma 4.20 implies that the Schützenberger involution indeed restricts to an action on  $\text{EYTab}(\lambda, \bar{\mu}\mu)$ , as required for Theorem 1.1 to be well-formulated.

*Proof.* In view of Proposition 3.30, we see that  $\xi(b)$  is a highest weight element of  $B_\lambda$  with content  $\bar{\gamma}\beta$ . The desired result then follows directly from Proposition 4.17.  $\square$

**Theorem 4.22.** *Let  $T \in B_\lambda$ . Then  $\varphi(\xi(T)) = (\xi(\text{Rect}(\varphi_1(T))), \xi(\varphi_0(T)))$ .*

*Remark 4.23.* To interpret the statement of Theorem 4.22, we understand  $\xi$  to denote the Schützenberger involution on  $\mathfrak{sl}_m$ -crystals as well as that on  $\mathfrak{sl}_{2m}$ -crystals.

*Proof.* Let  $T \in B_\lambda$ , and let  $\mathcal{C}$  be the connected component of  $B_\lambda$  containing  $T$ . Let  $b$  be the unique highest weight element of  $\mathcal{C}$ . Our proof is by induction on the length of the shortest path in the crystal from  $b$  to  $T$ . We see from Lemma 4.20 that the desired equality holds for the base case  $T = b$ .

For the inductive step, it suffices to show that if  $1 \leq i \leq m-1$  or  $m+1 \leq i \leq 2m-1$ , then

$$\varphi(\xi(T)) = (\xi(\text{Rect}(\varphi_1(T))), \xi(\varphi_0(T)))$$

implies

$$\varphi(\xi(f_i T)) = (\xi(\text{Rect}(\varphi_1(f_i T))), \xi(\varphi_0(f_i T))).$$

Note that

$$\begin{aligned} \varphi(\xi(f_i T)) &= \varphi(e_{2m-i} \xi(T)) = e_{2m-i} \varphi(\xi(T)) \\ &= e_{2m-i} (\xi(\text{Rect}(\varphi_1(T))), \xi(\varphi_0(T))). \end{aligned}$$

If  $1 \leq i \leq m-1$ , then

$$\begin{aligned} \varphi(\xi(f_i T)) &= (\xi(\text{Rect}(\varphi_1(T))), e_{m-i} \xi(\varphi_0(T))) \\ &= (\xi(\text{Rect}(\varphi_1(T))), \xi(f_i \varphi_0(T))) \\ &= (\xi(\text{Rect}(\varphi_1(f_i T))), \xi(\varphi_0(f_i T))). \end{aligned}$$

If  $m+1 \leq i \leq 2m-1$ , then

$$\begin{aligned} \varphi(\xi(f_i T)) &= (e_{2m-i} \xi(\text{Rect}(\varphi_1(T))), \xi(\varphi_0(T))) \\ &= (\xi(f_{i-m}(\text{Rect}(\varphi_1(T)))), \xi(\varphi_0(T))) \\ &= (\xi(\text{Rect}(\varphi_1(f_i T))), \xi(\varphi_0(f_i T))). \end{aligned}$$

□

**Corollary 4.24.** *Let  $\mathcal{C}$  be a connected component of the  $(\mathfrak{sl}_m \oplus \mathfrak{sl}_m)$ -crystal  $B_\lambda$ , and let  $b$  be the unique highest weight element of  $\mathcal{C}$ . If  $\xi(b) \neq b$ , then  $\{T \in \mathcal{C} : \xi(T) = T\}$  is empty. Otherwise, there exists a partition  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$  such that  $\varphi(b) = (b_{\bar{\beta}}, b_\beta)$ , and the isomorphism of crystals  $\varphi|_{\mathcal{C}} : \mathcal{C} \xrightarrow{\sim} B_{(\bar{\beta}, \beta)}$  restricts to a bijection of sets*

$$\{T \in \mathcal{C} : \xi(T) = T\} \xrightarrow{\sim} \{(U, U') \in B_{(\bar{\beta}, \beta)} : \xi(U) = U'\}.$$

We proceed to the proof of Theorem 1.1 itself. Let  $T'$  be an arbitrary standard tableau of shape  $\lambda'$ . We compute the character  $\chi$  of the  $GL_{2m}(\mathbb{C})$ -representation  $V_{T', 2m}$  at the element  $w_0 \cdot \text{diag}(x_1, x_2, \dots, x_m, x_m, \dots, x_2, x_1)$ . Note that

$$\begin{aligned} &\chi(w_0 \cdot \text{diag}(x_1, x_2, \dots, x_m, x_m, \dots, x_2, x_1)) \\ &= (-1)^{v(\lambda)} \cdot \sum_{T \in B_\lambda : \xi(T) = T} x_1^{-2T_1} x_2^{-2T_2} \dots x_m^{-2T_m} \\ &= (-1)^{v(\lambda)} \cdot \sum_{\theta \vdash |\lambda|/2} |\text{EYTab}^\xi(\lambda, \bar{\theta}\theta)| \cdot s_\theta(x_1^{-2}, x_2^{-2}, \dots, x_m^{-2}), \end{aligned}$$

where the first equality follows from Theorems 4.6 and 4.13, and the second equality follows from Corollary 4.24.

However,

$$w_0 \cdot \text{diag}(x_1, x_2, \dots, x_m, x_m, \dots, x_2, x_1)$$

is conjugate to

$$\text{diag}(x_1, x_2, \dots, x_m, -x_m, \dots, -x_2, -x_1).$$

Since  $\chi: GL_{2m}(\mathbb{C}) \rightarrow \mathbb{C}$  is a class function, we see that

$$\begin{aligned} & \chi(w_0 \cdot \text{diag}(x_1, x_2, \dots, x_m, x_m, \dots, x_2, x_1)) \\ &= \chi(\text{diag}(x_1, x_2, \dots, x_m, -x_m, \dots, -x_2, -x_1)) \\ &= s_\lambda(x_1^{-1}, x_2^{-1}, \dots, x_m^{-1}, -x_m^{-1}, \dots, -x_2^{-1}, -x_1^{-1}) \\ &= (-1)^{v(\lambda)} \sum_D x_1^{-2D_1} x_2^{-2D_2} \dots x_m^{-2D_m}, \end{aligned}$$

where the sum ranges over all semistandard domino tableaux of shape  $\lambda$  with entries in  $\{1, 2, \dots, m\}$ . (Here the second equality follows from Theorem 4.6, and the third equality follows from Remark 3.2 of Stembridge [27].)

By Theorem 2.28,

$$\sum_D x_1^{-2D_1} x_2^{-2D_2} \dots x_m^{-2D_m} = \epsilon_2(\lambda) \cdot \phi_2(s_\lambda)(x_1^{-2}, x_2^{-2}, \dots, x_m^{-2}).$$

Expanding via Corollary 2.21, we find that

$$\begin{aligned} \phi_2(s_\lambda)(x_1^{-2}, x_2^{-2}, \dots, x_m^{-2}) &= \sum_{\theta \vdash |\lambda|/2} \langle \phi_2(s_\lambda), s_\theta \rangle s_\theta(x_1^{-2}, x_2^{-2}, \dots, x_m^{-2}) \\ &= \sum_{\theta \vdash |\lambda|/2} \langle s_\lambda, p_2 \circ s_\theta \rangle s_\theta(x_1^{-2}, x_2^{-2}, \dots, x_m^{-2}). \end{aligned}$$

Identifying the coefficients of  $s_\mu(x_1^{-2}, x_2^{-2}, \dots, x_m^{-2})$  in our two expressions for  $\chi(w_0 \cdot \text{diag}(x_1, x_2, \dots, x_m, x_m, \dots, x_2, x_1))$  in accordance with Corollary 2.22, we may conclude that

$$|\text{EYTab}^\xi(\lambda, \bar{\mu}\mu)| = \epsilon_2(\lambda) \cdot \langle s_\lambda, p_2 \circ s_\mu \rangle.$$

□

**4.3. Proof of Theorem 1.2.** Let  $\lambda$  be a rectangular partition with  $mn$  parts. Suppose that  $n$  divides  $|\lambda|$ , and let  $\mu = (\mu_1, \mu_2, \dots, \mu_m)$  be a partition of  $|\lambda|/n$ . Let  $B_\lambda$  be the set of semistandard tableaux of shape  $\lambda$ , endowed with a  $\mathfrak{sl}_{mn}$ -crystal structure by the sets of raising and lowering operators  $\{e_i\}_{i=1}^{mn-1}$  and  $\{f_i\}_{i=1}^{mn-1}$ , respectively. The key to our proof is the assignment of an  $(\mathfrak{sl}_m^{\oplus n})$ -crystal structure to  $B_\lambda$  that allows us to inspect the action of  $j$  on its connected components. This process provides a combinatorial model for the decomposition into irreducible components of the restriction to  $GL_m(\mathbb{C})^{\times n}$  of the irreducible  $GL_{mn}(\mathbb{C})$ -representation with highest weight  $\lambda$ , which underlies our character evaluation.

Recall that we chose  $\{E_1 - E_2, E_2 - E_3, \dots, E_{mn-1} - E_{mn}\}$  as the set of simple roots for  $\mathfrak{sl}_{mn}$ . Here we choose  $\bigcup_{k=0}^{n-1} \{E_{km+1} - E_{km+2}, E_{km+2} - E_{km+3}, \dots, E_{(k+1)m-1} - E_{(k+1)m}\}$  as the set of simple roots for  $\mathfrak{sl}_m^{\oplus n}$ .

**Proposition 4.25.** *The set  $B_\lambda$  equipped with the map  $\text{wt}$ , the set of raising operators  $\bigcup_{k=0}^{n-1} \{e_{km+1}, e_{km+2}, \dots, e_{(k+1)m-1}\}$ , and the set of lowering operators  $\bigcup_{k=0}^{n-1} \{f_{km+1}, f_{km+2}, \dots, f_{(k+1)m-1}\}$  is a regular  $(\mathfrak{sl}_m^{\oplus n})$ -crystal.*

*Proof.* It is a simple matter to verify that the conditions of Definition 3.1 hold for  $\mathfrak{g} = \mathfrak{sl}_m^{\oplus n}$  with the indicated choice of simple roots. Hence  $B_\lambda$  is an  $(\mathfrak{sl}_m^{\oplus n})$ -crystal. To establish regularity, note that drawing any two operators from distinct sets among the collection

$$\{\{e_{km+1}, e_{km+2}, \dots, e_{(k+1)m-1}, f_{km+1}, f_{km+2}, \dots, f_{(k+1)m-1}\}\}_{k=0}^{n-1}$$

yields a commuting pair, so it suffices to show conditions (iv) - (vib) of Definition 3.3 hold for all  $i \neq j$  for which there exists  $0 \leq k \leq n-1$  such that  $km+1 \leq i, j \leq (k+1)m-1$ . This is immediate because  $B_\lambda$  is regular as an  $\mathfrak{sl}_{mn}$ -crystal.  $\square$

Each tableau in  $B_\lambda$  is made up of  $n$  “subtableaux,” the  $(k+1)^{\text{th}}$  among which has entries in  $\{km+1, km+2, \dots, (k+1)m-1\}$  for all  $0 \leq k \leq n-1$ . These subtableaux do not interact with each other under any of the raising and lowering  $(\mathfrak{sl}_m^{\oplus n})$ -crystal operators, so it is worthwhile to consider them independently.

**Definition 4.26.** For all tableaux  $T \in B_\lambda$  and  $0 \leq k \leq n-1$ , let  $\varphi_k(T)$  be the tableau obtained from  $T$  by removing each box with an entry not in  $\{km+1, km+2, \dots, (k+1)m-1\}$ , and reducing modulo  $m$  the entry in each remaining box, so that the entries of  $\varphi_k(T)$  are among  $1, 2, \dots, m$ . Let  $\varphi(T)$  be the ordered  $n$ -tuple of tableaux  $(\varphi_0(T), \text{Rect}(\varphi_1(T)), \dots, \text{Rect}(\varphi_{n-1}(T)))$ .

**Proposition 4.27.** *Let  $\mathcal{C}$  be a connected component of the  $(\mathfrak{sl}_m^{\oplus n})$ -crystal  $B_\lambda$ . Then  $\mathcal{C}$  is a highest weight crystal. Furthermore, if  $b$  is the unique highest weight element of  $\mathcal{C}$ , then there exist partitions  $\beta_0, \beta_1, \dots, \beta_{n-1}$ , each with  $m$  parts, such that  $b$  is of content  $\beta_0\beta_1 \cdots \beta_{n-1}$  and  $\varphi(b) = (b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})$ .*

*Proof.* Since  $\mathcal{C}$  is a regular connected crystal, it follows from Proposition 3.7 that  $\mathcal{C}$  is a highest weight crystal. Let  $b$  be the unique highest weight element of  $\mathcal{C}$ . Recall that  $b$  is Yamanouchi in  $\{km+1, km+2, \dots, (k+1)m\}$  for all  $0 \leq k \leq n-1$ . From Proposition 3.40 (in view of Corollary 3.16), we see that, if  $0 \leq k \leq n-1$ , then there exists a partition  $\beta_k = (\beta_{k,1}, \beta_{k,2}, \dots, \beta_{k,m})$  such that  $\text{Rect}(\varphi_k(T)) = b_{\beta_k}$ .  $\square$

**Proposition 4.28.** *Let  $\beta_0, \beta_1, \dots, \beta_{n-1}$  be partitions, each with  $m$  parts. Equip the set  $B_{(\beta_0, \beta_1, \dots, \beta_{n-1})} := B_{\beta_0} \times B_{\beta_1} \times \cdots \times B_{\beta_{n-1}}$  with the map  $\text{wt} \times \text{wt} \times \cdots \times \text{wt}$ . For all  $1 \leq i \leq m-1$  and  $0 \leq k \leq n-1$ , let  $e_{km+i}$  and  $f_{km+i}$  act as the  $\mathfrak{sl}_m$ -crystal operators  $e_i$  and  $f_i$ , respectively, on  $B_{\beta_k}$  and as the identity on  $B_{\beta_j}$  for all  $j \neq k$ . Then  $B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$ , together with the set*

of raising operators  $\bigcup_{k=0}^{n-1} \{e_{km+1}, e_{km+2}, \dots, e_{(k+1)m-1}\}$  and the set of lowering operators  $\bigcup_{k=0}^{n-1} \{f_{km+1}, f_{km+2}, \dots, f_{(k+1)m-1}\}$ , is a regular connected  $(\mathfrak{sl}_m^{\oplus n})$ -crystal with unique highest weight element  $(b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})$ .

*Proof.* It is apparent that  $B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$  is an  $(\mathfrak{sl}_m^{\oplus n})$ -crystal. By the reasoning in the proof of Proposition 4.25, to establish regularity it suffices to show conditions (iv) - (vib) of Definition 3.3 hold for all  $i \neq j$  for which there exists  $0 \leq k \leq n-1$  such that  $km+1 \leq i, j \leq (k+1)m-1$ . This is immediate because  $B_{\beta_k}$  is regular as an  $\mathfrak{sl}_m$ -crystal for all  $0 \leq k \leq n-1$ .

The claim that  $(b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})$  is the unique highest weight element of  $B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$  is a direct consequence of Proposition 3.40.  $\square$

Thus, if  $\mathcal{C}$  is a connected component of  $B_\lambda$ , there exist partitions  $\beta_k$ ,  $0 \leq k \leq n-1$ , for which the unique highest weight element of  $\mathcal{C}$  corresponds to that of  $B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$ . In fact, the two crystals are structurally identical.

**Theorem 4.29.** *Let  $\mathcal{C}$  be a connected component of the  $(\mathfrak{sl}_m^{\oplus n})$ -crystal  $B_\lambda$ . Let  $b$  be the unique highest weight element of  $\mathcal{C}$ , and let  $\beta_0, \beta_1, \dots, \beta_{n-1}$  be partitions, each with  $m$  parts, for which  $\varphi(b) = (b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})$ . Then  $\varphi$  restricts to an isomorphism of crystals  $\mathcal{C} \xrightarrow{\sim} B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$ .*

*Proof.* Since *jeu-de-taquin* slides commute with crystal operators (cf. Proposition 3.15), it should be clear that  $\varphi|_{\mathcal{C}}: \mathcal{C} \rightarrow B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$  is a morphism of crystals. Furthermore, the equality  $\phi_i(b) = \phi_i(b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})$  holds for all  $1 \leq i \leq m-1$  by definition of  $\varphi_0$ , and it holds for all  $1 \leq k \leq n-1$  and  $km+1 \leq i \leq (k+1)m-1$  by definition of  $\varphi_k$  in view of Corollary 3.16. Thus, Proposition 3.8 tells us that  $\mathcal{C}$  and  $B_{(b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})}$  are isomorphic. A morphism of crystals  $\mathcal{C} \rightarrow B_{(\beta_0, \beta_1, \dots, \beta_{n-1})}$  is uniquely determined by its image at  $b$ , so we may conclude that  $\varphi|_{\mathcal{C}}$  is an isomorphism.  $\square$

We turn our attention now to the action of  $j^d$  for  $d$  dividing  $n$ , first on the highest weight elements of the  $(\mathfrak{sl}_m^{\oplus n})$ -crystal  $B_\lambda$ , and then on all its tableaux.

**Lemma 4.30.** *Let  $b$  be a highest weight element of  $B_\lambda$ , and let  $\beta_0, \beta_1, \dots, \beta_{n-1}$  be partitions, each with  $m$  parts, such that  $\varphi(b) = (b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-1}})$ . Then  $\varphi(j^d(b)) = (b_{\beta_{n-d}}, b_{\beta_{n-d+1}}, \dots, b_{\beta_{n-1}}, b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{n-d-1}})$ .*

*Remark 4.31.* If  $\mu$  is a partition with  $m$  parts, then  $\text{PYTab}(\lambda, \mu^n)$  is the set of highest weight elements of  $B_\lambda$  with content  $\mu^n$ . Thus, Lemma 4.30 implies that  $j = \text{pr}^m$  indeed restricts to an action on  $\text{PYTab}(\lambda, \mu^n)$ , as required for Theorem 1.2 to be well-formulated.

*Proof.* Recall from Theorem 3.25 that  $j^d(b) = j^{-(n-d)}(b)$ . In view of Proposition 3.20, we see that  $e_{km+1}, e_{km+2}, \dots, e_{(k+1)m-1}$  all vanish at  $j^d(b)$  for  $d \leq k \leq n-1$ . Rewriting property (ii) in Proposition 3.20 as  $\text{pr}^{-1}(e_{i+1} \cdot T) = e_i \cdot \text{pr}^{-1}(T)$ , we see that  $e_{km+1}, e_{km+2}, \dots, e_{(k+1)m-1}$  all vanish at  $j^{-(n-d)}(b)$  for  $0 \leq k \leq d-1$ . Thus,  $j^d(b)$  is a highest weight element of

$B_\lambda$ , and, from property (i) in Proposition 3.20, we see that  $j^d(b)$  is of content  $b_{\beta_{n-d}}b_{\beta_{n-d+1}} \cdots b_{\beta_{n-1}}b_{\beta_0}b_{\beta_1} \cdots b_{\beta_{n-d-1}}$ . The desired result then follows directly from Proposition 4.27.  $\square$

**Theorem 4.32.** *Let  $T \in B_\lambda$ . Then  $\varphi(j^d(T))$*

$$= (\text{Rect}(\varphi_{n-d}(T)), \dots, \text{Rect}(\varphi_{n-1}(T)), \varphi_0(T), \dots, \text{Rect}(\varphi_{n-d-1}(T))).$$

*Proof.* Let  $T \in B_\lambda$ , and let  $\mathcal{C}$  be the connected component of  $B_\lambda$  containing  $T$ . Let  $b$  be the unique highest weight element of  $\mathcal{C}$ . Our proof is by induction on the length of the shortest path in the crystal from  $b$  to  $T$ . We see from Lemma 4.30 that the desired equality holds for the base case  $T = b$ .

For the inductive step, it suffices to show that if  $0 \leq k \leq n-1$  and  $km+1 \leq i \leq (k+1)m-1$ , then

$$\varphi(j^d(T)) = (\text{Rect}(\varphi_{n-d}(T)), \dots, \varphi_0(T), \dots, \text{Rect}(\varphi_{n-d-1}(T)))$$

implies

$$\varphi(j^d(f_i T)) = (\text{Rect}(\varphi_{n-d}(f_i T)), \dots, \varphi_0(f_i T), \dots, \text{Rect}(\varphi_{n-d-1}(f_i T))).$$

Note that if  $0 \leq k \leq n-d-1$ , then

$$\begin{aligned} \varphi(j^d(f_i T)) &= \varphi(f_{i+dm} j^d(T)) = f_{i+dm} \varphi(j^d(T)) \\ &= f_{i+dm} (\text{Rect}(\varphi_{n-d}(T)), \dots, \varphi_0(T), \dots, \text{Rect}(\varphi_{n-d-1}(T))) \\ &= (\text{Rect}(\varphi_{n-d}(f_i T)), \dots, \varphi_0(f_i T), \dots, \text{Rect}(\varphi_{n-d-1}(f_i T))). \end{aligned}$$

If  $n-d \leq k \leq n-1$ , then

$$\begin{aligned} \varphi(j^d(f_i T)) &= \varphi(f_{i-(n-d)m} j^d(T)) = f_{i-(n-d)m} \varphi(j^d(T)) \\ &= f_{i-(n-d)m} (\text{Rect}(\varphi_{n-d}(T)), \dots, \varphi_0(T), \dots, \text{Rect}(\varphi_{n-d-1}(T))) \\ &= (\text{Rect}(\varphi_{n-d}(f_i T)), \dots, \varphi_0(f_i T), \dots, \text{Rect}(\varphi_{n-d-1}(f_i T))). \end{aligned}$$

$\square$

**Corollary 4.33.** *Let  $\mathcal{C}$  be a connected component of the  $(\mathfrak{sl}_m^{\oplus n})$ -crystal  $B_\lambda$ , and let  $b$  be the unique highest weight element of  $\mathcal{C}$ . If  $j^d(b) \neq b$ , then  $\{T \in \mathcal{C} : j^d(T) = T\}$  is empty. Otherwise, there exist  $d$  partitions  $\beta_0, \beta_1, \dots, \beta_d$ , each with  $m$  parts, such that  $\varphi(b) = ((b_{\beta_0}, b_{\beta_1}, \dots, b_{\beta_{d-1}})^{(n/d)})$ , and the isomorphism of crystals  $\mathcal{C} \xrightarrow{\sim} B_{((\beta_0, \beta_1, \dots, \beta_d)^{(n/d)})}$  restricts to a bijection of sets*

$$\begin{aligned} &\{T \in \mathcal{C} : j^d(T) = T\} \xrightarrow{\sim} \\ &\{(U_0, U_1, \dots, U_{n-1}) \in B_{((\beta_0, \beta_1, \dots, \beta_d)^{(n/d)})} \\ &\text{such that } U_j = U_{j'} \text{ for all } j \cong j' \pmod{d}\}. \end{aligned}$$

We proceed to the proof of Theorem 1.2 itself. For all  $T \in B_\lambda$ , let

$$(T_{1,1}, T_{1,2}, \dots, T_{1,m}, T_{2,1}, T_{2,2}, \dots, T_{2,m}, \dots, T_{n,1}, T_{n,2}, \dots, T_{n,m})$$



be the content of  $T$ , and write  $T_i$  for the composition  $(T_{i,1}, T_{i,2}, \dots, T_{i,m})$  for all  $1 \leq i \leq n$ . Let  $T'$  be an arbitrary standard tableau of shape  $\lambda'$ , and set  $b := \lambda'_1$ . Let

$$\{y_{1,1}, y_{1,2}, \dots, y_{1,m}\}, \{y_{2,1}, y_{2,2}, \dots, y_{2,m}\}, \dots, \{y_{d,1}, y_{d,2}, \dots, y_{d,m}\}$$

be a collection of  $d$  variable sets denoted by  $y_1, y_2, \dots, y_d$ , respectively. By abuse of notation, let the corresponding diagonal matrices

$$\text{diag}(y_{1,1}, y_{1,2}, \dots, y_{1,m}), \text{diag}(y_{2,1}, y_{2,2}, \dots, y_{2,m}), \dots, \text{diag}(y_{d,1}, y_{d,2}, \dots, y_{d,m})$$

be denoted by  $y_1, y_2, \dots, y_d$ , respectively, as well. We compute the character  $\chi$  of the  $GL_{mn}(\mathbb{C})$ -representation  $V_{T',mn}$  at the element

$$c_{mn}^{md} \cdot \text{diag}(y_1, y_2, \dots, y_d, y_1, y_2, \dots, y_d, \dots, y_1, y_2, \dots, y_d).$$

Let  $\eta$  be a composition of  $|\lambda|$  with  $mn$  parts, and let  $U'$  be a row-strict tableau of shape  $\lambda'$  and content  $\eta$ . From Theorem 4.14, we see that

$$c_{mn}^{md} \cdot I_\eta(U') = (-1)^{(\eta_{m(n-d)+1} + \eta_{m(n-d)+2} + \dots + \eta_{mn})(b-1)} \cdot I_{c_{mn}^{md}, \eta}(\text{pr}^{md}(U')).$$

If  $\eta = c_{mn}^{md} \cdot \eta$ , then

$$\eta_{m(n-d)+1} + \eta_{m(n-d)+2} + \dots + \eta_{mn} = \frac{d}{n} \cdot |\eta| = \frac{d}{n} \cdot |\lambda|,$$

and

$$c_{mn}^{md} \cdot I_\eta(U') = (-1)^{\frac{d}{n} \cdot |\lambda|(b-1)} \cdot I_\eta(\text{pr}^{md}(U')) = \zeta^{dv(\lambda)} \cdot I_\eta(\text{pr}^{md}(U')),$$

where  $\zeta$  is a primitive  $n^{\text{th}}$  root of unity.

Note that

$$\begin{aligned} & \chi(c_{mn}^{md} \cdot \text{diag}(y_1, y_2, \dots, y_d, y_1, y_2, \dots, y_d, \dots, y_1, y_2, \dots, y_d)) \\ &= \zeta^{dv(\lambda)} \cdot \sum_{T \in B_\lambda: j^d(T)=T} y_1^{-\frac{n}{d} \cdot T_1} y_2^{-\frac{n}{d} \cdot T_2} \dots y_d^{-\frac{n}{d} \cdot T_d} \\ &= \zeta^{dv(\lambda)} \cdot \sum |\text{PYTab}^{j^d}(\lambda, (\theta_1 \theta_2 \dots \theta_d)^{\frac{n}{d}})| \cdot s_{\theta_1} \left( y_1^{-\frac{n}{d}} \right) s_{\theta_2} \left( y_2^{-\frac{n}{d}} \right) \dots s_{\theta_d} \left( y_d^{-\frac{n}{d}} \right), \end{aligned}$$

where the sum ranges over all  $d$ -tuples of partitions  $(\theta_1, \theta_2, \dots, \theta_d)$  such that  $|\theta_1| = |\theta_2| = \dots = |\theta_d| = |\lambda|/n$ . (The first equality follows from Theorems 4.6 and 4.14, and the second equality follows from Corollary 4.33.)

However,

$$c_{mn}^{md} \cdot \text{diag}(y_1, y_2, \dots, y_d, y_1, y_2, \dots, y_d, \dots, y_1, y_2, \dots, y_d)$$

is conjugate to

$$\text{diag}(y_1, y_2, \dots, y_d, \zeta^d y_1, \zeta^d y_2, \dots, \zeta^d y_d, \dots, \zeta^{n-d} y_1, \zeta^{n-d} y_2, \dots, \zeta^{n-d} y_d).$$

Since  $\chi: GL_{mn}(\mathbb{C}) \rightarrow \mathbb{C}$  is a class function, we see that

$$\begin{aligned} & \chi(c_{mn}^{md} \cdot \text{diag}(y_1, y_2, \dots, y_d, y_1, y_2, \dots, y_d, \dots, y_1, y_2, \dots, y_d)) \\ &= \chi(\text{diag}(y_1, y_2, \dots, y_d, \zeta^d y_1, \zeta^d y_2, \dots, \zeta^d y_d, \dots, \zeta^{n-d} y_1, \zeta^{n-d} y_2, \dots, \zeta^{n-d} y_d)) \\ &= s_\lambda(y_1^{-1}, \dots, y_d^{-1}, \zeta^d y_1^{-1}, \dots, \zeta^d y_d^{-1}, \dots, \zeta^{n-d} y_1^{-1}, \dots, \zeta^{n-d} y_d^{-1}) \\ &= \zeta^{dv(\lambda)} \sum_R y_1^{-\frac{n}{d} \cdot R_1} y_2^{-\frac{n}{d} \cdot R_2} \dots y_d^{-\frac{n}{d} \cdot R_d}, \end{aligned}$$

where the sum ranges over all semistandard  $(n/d)$ -ribbon tableaux of shape  $\lambda$  with entries in  $\{1, 2, \dots, md\}$ . (For all such  $(n/d)$ -ribbon tableaux  $R$ , the content of  $R$  is denoted by

$$(R_{1,1}, R_{1,2}, \dots, R_{1,m}, R_{2,1}, R_{2,2}, \dots, R_{2,m}, \dots, R_{d,1}, R_{d,2}, \dots, R_{d,m}),$$

and, for all  $1 \leq i \leq d$ , the composition  $(R_{i,1}, R_{i,2}, \dots, R_{i,m})$  is denoted by  $R_i$ .) Here the second equality follows from Theorem 4.6, and the third equality follows from Lemma 6.2 of Rhoades [20].

By Theorem 2.28,

$$\sum_R y_1^{-\frac{n}{d} \cdot R_1} y_2^{-\frac{n}{d} \cdot R_2} \dots y_d^{-\frac{n}{d} \cdot R_d} = \epsilon_{n/d}(\lambda) \cdot \phi_{n/d}(s_\lambda) \left( y_1^{-\frac{n}{d}}, y_2^{-\frac{n}{d}}, \dots, y_d^{-\frac{n}{d}} \right).$$

Expanding via Corollary 2.21, we find that

$$\begin{aligned} & \phi_{n/d}(s_\lambda) \left( y_1^{-\frac{n}{d}}, y_2^{-\frac{n}{d}}, \dots, y_d^{-\frac{n}{d}} \right) \\ &= \sum \langle \phi_{n/d}(s_\lambda), s_{\theta_1} s_{\theta_2} \dots s_{\theta_d} \rangle s_{\theta_1} \left( y_1^{-\frac{n}{d}} \right) s_{\theta_2} \left( y_2^{-\frac{n}{d}} \right) \dots s_{\theta_d} \left( y_d^{-\frac{n}{d}} \right) \\ &= \sum \langle s_\lambda, p_{n/d} \circ (s_{\theta_1} s_{\theta_2} \dots s_{\theta_d}) \rangle s_{\theta_1} \left( y_1^{-\frac{n}{d}} \right) s_{\theta_2} \left( y_2^{-\frac{n}{d}} \right) \dots s_{\theta_d} \left( y_d^{-\frac{n}{d}} \right), \end{aligned}$$

where again the sums range over all  $d$ -tuples of partitions  $(\theta_1, \theta_2, \dots, \theta_d)$  such that  $|\theta_1| = |\theta_2| = \dots = |\theta_d| = |\lambda|/n$ .

Note that  $p_{n/d} \circ (s_{\theta_1} s_{\theta_2} \dots s_{\theta_d}) = (s_{\theta_1} s_{\theta_2} \dots s_{\theta_d}) \circ p_{n/d}$  in view of Proposition 2.27. It follows from Equation 6.4 in Macdonald [18], Chapter 1, that  $(g_1 g_2) \circ h = (g_1 \circ h)(g_2 \circ h)$  for all symmetric functions  $g_1, g_2, h \in \Lambda$ . Thus, we see inductively that  $(s_{\theta_1} s_{\theta_2} \dots s_{\theta_d}) \circ p_{n/d} = (s_{\theta_1} \circ p_{n/d})(s_{\theta_2} \circ p_{n/d}) \dots (s_{\theta_d} \circ p_{n/d})$ . Invoking Proposition 2.27 again, we find that  $p_{n/d} \circ (s_{\theta_1} s_{\theta_2} \dots s_{\theta_d}) = (p_{n/d} \circ s_{\theta_1})(p_{n/d} \circ s_{\theta_2}) \dots (p_{n/d} \circ s_{\theta_d})$ .

Thus, identifying the coefficients of  $s_\mu(y_1^{-\frac{n}{d}}) s_\mu(y_2^{-\frac{n}{d}}) \dots s_\mu(y_d^{-\frac{n}{d}})$  in our two expressions for

$$\chi(c_{mn}^{md} \cdot \text{diag}(y_1, y_2, \dots, y_d, y_1, y_2, \dots, y_d, \dots, y_1, y_2, \dots, y_d))$$

in accordance with Corollary 2.22, we may conclude that

$$|\text{PYTab}^{j^d}(\lambda, \mu^n)| = \epsilon_{n/d}(\lambda) \cdot \langle s_\lambda, p_{n/d}^d \circ s_\mu \rangle.$$

□

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